Pontificia Universidad Católica del Perú

Escuela de Posgrado

Curvatura y fibrados principales sobre el círculo (Curvature and principal S^1-bundles)

Tesis para optar el Grado de Magister en Matemáticas

Joe Moises Lope Vicente

Asesor

Dr. Jaime Cuadros

Jurado

Dr. Alfredo Poirier

Dr. Roland Rabanal

Lima - Perú

Agosto - 2018

Curvatura y fibrados principales sobre el círculo
(Curvature and principal S^1-bundles)

Joe Lope Vicente [1]

Tesis presentada a consideración del cuerpo docente de la escuela de posgrado de la PUCP como parte de los requisitos para obtener el grado académico de Magíster en Matemáticas.

Miembros de Jurado:

Dr. Jaime Cuadros (Asesor)

Dr Alfredo Poirier (presidente)

Dr. Roland Rabanal

Lima - Perú
Agosto - 2018

[1] Agradecimiento a la D.G.I. por el financiamiento ecónomico de esta tesis a través del proyecto 2015-1-0048.

Abstract

The aim of this thesis is to study in detail the work of S. Kobayashi on the Riemannian geometry on principal S^1-bundles. To be more precise, we explain how to obtain metrics with constant scalar curvature on these bundles. The method that we use is based in [18].

The basic idea behind Kobayashi's construction is to slightly deform the Hopf fibration $S^1 \hookrightarrow S^{2n+1} \longrightarrow \mathbb{CP}^n$ in a such a way that the corresponding sectional curvatures are not far from the produced by the standard metrics on the sphere and the complex projective space on the Hopf fibration. This deformations can be controlled applying the notions of Riemaniann and Kahlerian pinching (see Chapter 3).

Furthermore, thanks to a technique developed by Hatakeyama in [14], it is possible to obtain less generic metrics but with a larger set of symmetries on the total space: Sasaki metrics. Actually, If one chooses as a base space a Kähler-Einstein manifold with positive scalar curvature one can obtain a Sasaki-Einstein metric.

Acknowledgements

A mis padres y a mis hermanos por su compañía y afecto durante esta etapa de mi vida.

A los profesores de la maestría en matemática de la escuela de posgrado de la PUCP. En especial a mi asesor de tesis, el profesor Jaime Cuadros, por haberme guiado durante el desarrollo de este trabajo, por su paciencia, y por su confianza. De igual forma a los profesores Roland Rabanal y Alfredo Poirier por las sugerencias hechas a mi trabajo, y por haberle dedicado su tiempo a la lectura del mismo. Asímismo al profesor Richard Gonzales, por sus consejos.

A mis amigos de la maestría por su apoyo, sus preguntas y sugerencias.

A la Dirección de Gestión de la Investigación de la PUCP, por haberme apoyado en la elaboración de esta tesis a través del proyecto 2015-1-0048.

. . . A mis padres.

Contents

Introduction

In this thesis we explain the construction of Einstein metrics on circle bundles over Kähler manifolds, following a technique due to S. Kobayashi which deals with a natural question: If one chooses an Einstein metric on the base space of a circle bundle, is it possible to lift this metric to the total space in order to obtain an Einstein metric as well, as achieved, for instance, on Hopf fibrations? Kobayashi's method reduces the possibilities of a positive answer to this question to a very concrete list of possibilities.

In the first chapter we survey some elementary tools that are used in the development of this work. This chapter covers some topics on complex geometry, Kählerian geometry, contact and symplectic geometry. In the second chapter we begin studying the connection and curvature forms on principal bundles. We end this chapter endowing the set of principal S^1-bundles with the structure of an abelian group, that is, we define the topological Picard group which can be identified with the integral second cohomology group of the base space.

In the third chapter we present the result established by S. Kobayashi in [18]. For that, we study the riemaniann metrics on the base and total space of a principal S^1-bundle, under mild conditions these metrics turn out to possess close related sectional curvatures, which are studied in much detail in this work. This permits certain control on the geometry of the fiber bundle as a whole. In the fourth chapter we construct an Einstein metric on the total space using the method given by S. Kobayashi in [18]. After rescaling the metric, one can establish the existence of Sasaki-Einstein metrics on the total space of circle bundles fibered over Kähler-Einstein manifolds with positive Einstein constant.

Chapter 1

Preliminaries

1.1 About Lie groups and the exponential map

Let G be a Lie group and M a smooth manifold. A *left action* of G on M is a smooth map

$$\begin{aligned} G \times M &\longrightarrow M \\ (g, m) &\longrightarrow gm \end{aligned}$$

such that $em = m$ and $(g_1g_2)m = g_1(g_2m)$, where e represents the identity of G. Analogously, a *right action* of G on M is defined as a smooth map

$$\begin{aligned} M \times G &\longrightarrow M \\ (m, g) &\longrightarrow mg, \end{aligned}$$

where $me = m$ and $m(g_1g_2) = (mg_1)g_2$. We will refer to left and right actions as *actions*.

Since G is a smooth manifold, we can define actions of G on itself using the left and the right multiplication. To be precise, these actions allow us to establish diffeomorphisms on G as follows: fixing $g \in G$, we define the diffeomorphisms L_g and R_g as $L_g(h) = gh$ and $R_g(h) = hg$, respectively.

In general, if G is a Lie group acting on a smooth manifold M, for each $g \in G$ we obtain diffeomorphisms φ_g on M defined by $\varphi_g(m) = gm$. The set $\{\varphi_g\}_{g\in tt}$ is a subgroup of Diff(M) under composition.

Definition 1.1.1. Let G be a Lie group acting on a manifold M. For each $m \in M$ the set

$$G_m = \{g \in G : gm = m\}$$

is called the *stabilizer* of m, and the set

$$Gm = \{gm \in M \; : g \in G\}$$

is called the *orbit* of m.

Remark 1.1.2.

(i) Clearly the stabilizer G_m of m is a Lie subgroup of G for all $m \in M$.

(ii) If G acts to the right on M and the stabilizer G_m is trivial for each $m \in M$, we say that G *acts freely* on M. For instance, this happens in principal bundles as we will soon see.

Recall that for each $g \in G$ we have defined diffeomorphisms L_g and R_g on M. We call *left invariant vector field* (respectively *right invariant vector field*) a vector field $X \in \mathfrak{X}(G)$ that verifies $(L_g)_*(X) = X$ (respectively $(R_g)_*(X) = X$).

Each left invariant vector field $X \in \mathfrak{g}$ is uniquely determined by its value $X(e)$ in T_eG. Thus, we can identify the space of left invariant vector fields with T_eG. We refer to both spaces as $\mathfrak{g}$.

Proposition 1.1.3. *If X and Y are left invariant vector fields, we have $[X, Y] \in \mathfrak{g}$. As a consequence we may regard $\mathfrak{g}$ as a Lie algebra.*

Proof. See Warner [30]. □

A differential form ω on G is called *left invariant* if $(L_g)^*\omega = \omega$ for all $g \in G$. We establish an isomorphism between the space of left invariant 1-forms and the dual space of T_eG through the assignment $\omega \mapsto \omega_e$. We refer to both spaces as $\mathfrak{g}^*$.

Remark 1.1.4. Let α be a left invariant 1-form on G. Then the map $\alpha(X)$ is constant for any $X \in \mathfrak{g}$ and we get $X(\alpha(Y)) = 0$, for all $X, Y \in \mathfrak{g}$. Therefore, applying the usual formula for 2-forms

$$d\omega(X, Y) = \frac{1}{2}\{X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])\},$$

for this α in particular we obtain

$$d\alpha(X, Y) = \frac{1}{2}\{X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])\} = -\frac{1}{2}\alpha([X, Y]),$$

known as the *Maurer-Cartan equation.*

A $\mathfrak{g}$-valued 1-form ω on G is called *left invariant* if $(L_g)^*\omega = \omega$. We define the *Maurer-Cartan form* as the $\mathfrak{g}$-valued 1-form $\theta_g = (L_{g^{-1}})_* : T_gG \to \mathfrak{g}$. Clearly θ is left invariant and satisfies $\theta_e = id_{T_e tt}$.

Let M be a smooth manifold. We can define a product on the space $\Lambda^1(M) \otimes \mathfrak{g}$ of $\mathfrak{g}$-valued 1-forms as follows. Suppose a basis $\{g_i\}$ of $\mathfrak{g}$ in given and α, β are two $\mathfrak{g}$-valued 1-forms on M. Then there exist 1-forms α^i, β^j such that

$$\alpha = \sum_i \alpha^i \otimes g_i \text{ and } \beta = \sum_j \beta^j \otimes g_j.$$

We define the product on $\Lambda^1(M) \otimes \mathfrak{g}$ as

$$[\alpha, \beta] = \sum_{i,j} \alpha^i \wedge \beta^j \otimes [g_i, g_j].$$

Obviously we have $[\alpha, \beta] \in \Lambda^2(M) \otimes \mathfrak{g}$. In general, for $\alpha = \sum_i \alpha^i \otimes g_i \in \Lambda^p(M) \otimes \mathfrak{g}$ and $\beta = \sum_j \beta^j \otimes g_j \in \Lambda^q(M) \otimes \mathfrak{g}$, the product of α and β is defined by

$$[\alpha, \beta] = \sum_{i,j} \alpha^i \wedge \beta^j \otimes [g_i, g_j] \in \Lambda^{p+q}(M) \otimes \mathfrak{g}.$$

When $\alpha = \sum_i \alpha^i \otimes g_i$ and $\beta = \sum_j \beta^j \otimes g_j$ are two $\mathfrak{g}$-valued left invariant 1-forms on G, and $X, Y \in \mathfrak{X}(G)$, we obtain

$$[\alpha(X), \beta(Y)] = \sum_{i,j} [\alpha^i(X)g_i, \beta^j(Y)g_j] = \sum_{i,j} \alpha^i(X)\beta^j(Y)[g_i, g_j].$$

The last equality holds due to the fact that $\alpha^i(X)$ and $\beta^j(Y)$ are constants. In a similar way we get

$$[\alpha(Y), \beta(X)] = \sum_{i,j} \alpha^i(Y)\beta^j(X)[g_i, g_j].$$

As a result we obtain

$$[\alpha, \beta](X, Y) = \frac{1}{2}\{[\alpha(X), \beta(Y)] - [\alpha(Y), \beta(X)]\}.$$

In particular, if we set $\alpha = \beta$ in the equation above, this yields

$$[\alpha, \alpha](X, Y) = [\alpha(X), \alpha(Y)]. \tag{1.1}$$

We can also define an exterior differentiation over $\Lambda^p(M) \otimes \mathfrak{g}$ via

$$\begin{aligned} d: \quad \Lambda^p(M) \otimes \mathfrak{g} &\longrightarrow \Lambda^{p+1}(M) \otimes \mathfrak{g} \\ \alpha \otimes \mathfrak{g} &\longrightarrow d(\alpha \otimes \mathfrak{g}) = d\alpha \otimes \mathfrak{g}. \end{aligned}$$

Take θ the Maurer-Cartan form written as $\theta = \sum_i \theta^i \otimes g_i$, where $\{g_i\}$ is a basis of $\mathfrak{g}$ and $\{\theta^i\}$ is a set of left invariant 1-forms on G. Then, using the Maurer-Cartan equation we arrive to

$$d\theta(X, Y) = \sum_i d\theta^i(X, Y)g_i = \sum_i -\frac{1}{2}\theta^i([X, Y])g_i = -\frac{1}{2}\theta([X, Y]).$$

Since $\theta_e = id_{\mathfrak{g}}$, we get

$$\theta([X, Y]) = [X, Y] = [\theta(X), \theta(Y)].$$

Taking the result above and Equation (1.1) together, we obtain

$$d\theta(X, Y) = -\frac{1}{2}\theta([X, Y]) = -\frac{1}{2}[\theta(X), \theta(Y)] = -\frac{1}{2}[\theta, \theta](X, Y).$$

In brief, θ is characterized by the relation

$$d\theta = -\frac{1}{2}[\theta, \theta],$$

known as the *Maurer-Cartan structure equation.*

Let $\mathfrak{g}$ be the Lie algebra of G. Fixing $X \in \mathfrak{g}$, each $g \in G$ determines the differential equation

$$\dot{\gamma} = X(\gamma), \qquad \gamma(0) = e.$$

By the existence and uniqueness of solutions of ODE's, there exists a unique solution $\gamma_X : \mathbb{R} \to G$ of the above equation. Notice that in the case of Lie groups, this solution is global. The elements of the 1-parameter group $\{\Phi_t\}$ generated by X are characterized by $\Phi_t(g) = g\gamma_X(t)$. Since γ_X is globally defined, each Φ_t is a diffeomorphism on M, and thus X is a complete vector field.

Remark 1.1.5. The above map γ_X verifies

1. $\gamma_X(t+s) = \gamma_X(t)\gamma_X(s)$,

2. $\gamma_{tX}(s) = \gamma_X(ts)$.

Definition 1.1.6. Let X be in $\mathfrak{g}$ and $\{\Phi_t\}$ be the 1-parameter group generated by X. The *exponential map* is defined as the smooth map

$$\begin{aligned} \exp\colon \quad \mathfrak{g} &\longrightarrow G \\ X &\longrightarrow \exp(X) = \Phi_1(e) = \gamma_X(1). \end{aligned}$$

By the above definition, we have $\gamma_X(t) = \exp(tX)$. Hence, if $\{\Phi_t\}$ is the flow associated to X, we have $\Phi_t(g) = g\exp(tX)$.

Next we state some properties of the exponential map.

Remark 1.1.7. We have the following identities for the exponential map

1. $\exp((s+t)X) = \exp(sX)\exp(tX)$,

2. $\exp(-tX) = (\exp(tX))^{-1}$,

3. $d_e \exp = Id_{\mathfrak{g}}$.

Let G be a Lie group and V a vector space. A linear representation of G on V is a homomorphism $\varphi : G \to GL(V)$ of Lie groups. In the following lines, we are going to obtain a representation of G on $\mathfrak{g}$. For each $g \in G$, we define the *conjugation map* $\sigma_g : G \to G$ by $\sigma_g(h) = ghg^{-1}$. Notice that $\sigma_g = L_g \circ R_{g^{-1}}$ and $\sigma_e = Id_{tt}$ hold. Since σ_g is a diffeomorphism on G, we get $(\sigma_g)_{*.e} \in GL(\mathfrak{g})$. Thus, each $g \in G$ defines an element $d_e\sigma_g$ in $GL(\mathfrak{g})$. Hence, we obtain a representation of G on $\mathfrak{g}$, called *adjoint representation*, given by

$$\begin{aligned} Ad\colon \quad G &\longrightarrow GL(\mathfrak{g}) \\ g &\longrightarrow Ad_g = (\sigma_g)_{*,e}. \end{aligned}$$

Lemma 1.1.8. *Let G be a Lie group. Then for each $g \in G$ the diagram*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "Ad_g"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "\sigma_g"] & G \end{tikzcd}$$

commutes.

Proof. See Warner [30]. □

Take $X \in \mathfrak{g}$ and $g \in G$. By Lemma 1.1.8, for the flow Φ_t generated by $Ad_g X$, we obtain

$$\Phi_t(a) = a \exp(tAd_g X) = ag \exp(tX)g^{-1}.$$

Thus, we get

$$Ad_g X = \frac{d}{dt}(g \exp(tX)g^{-1})|_{t=0} \quad .$$

In case G were a matrix group, this translates into

$$Ad_X Y = \frac{d}{dt}(X \exp(tY)X^{-1}) \quad |_{t=0} = X \frac{d}{dt}(\exp(tY)) \,|_{t=0} X^{-1} = XYX^{-1}.$$

Proposition 1.1.9. *For θ the Maurer-Cartan form, we have*

$$R_g^*\theta = Ad_{g^{-1}} \circ \theta.$$

Proof. In fact, for $u \in T_h G$, we get

$$\begin{aligned}(R_g^*\theta)_h(u) = \theta_{hg}((R_g)_*(u)) &= (L_{(hg)^{-1}})_* \circ (R_g)_*(u)\\ &= (L_{g^{-1}})_* \circ (L_{h^{-1}})_*(R_g)_*(u)\\ &= Ad_{g^{-1}} \circ (L_{h^{-1}})_*(u)\\ &= Ad_{g^{-1}} \circ \theta_h(u).\end{aligned}$$

□

1.2 Sheaf cohomology

Let (X, τ) be a topological space. A *presheaf of abelian groups* on X consists of a pair (F, ρ) where $\mathsf{F} = (\mathsf{F}(U))_{U\in\tau}$ is a collection of abelian groups and $\rho = (\rho_{U,V})_{U,V \in\tau, V \subset U}$ is a collection of group homomorphisms defined as $\rho_{U,V} : \mathsf{F}(U) \rightarrowtail \mathsf{F}(V)$, which verify the following properties

(i) $\rho_{U,U} = id_U$;

(ii) $\rho_{V,W} \circ \rho_{U,V} = \rho_{U,W}$, where $U, V, W \in \tau$ and $W \subset V \subset U$.

The homomorphisms $\rho_{U,V}$ are called *restriction maps.*

Remark 1.2.1. Given the group homomorphism $\rho_{U,V} : \mathsf{F}(U) \to \mathsf{F}(V)$, for any element $f \in \mathsf{F}(U)$ we denote $\rho_{U,V}(f)$ as $f|_V$,

Example 1.2.2. Let X be a topological space and A an abelian group where we impose the discrete topology. For each open $U \subset X$, we denote by $\mathsf{F}(U)$ the set of all continuous functions $f : U \to A$. It is easy to see that $\mathsf{F}(U)$ has an abelian group structure. Moreover, we can define the restriction maps as

$$\begin{aligned} \rho_{U,V}: \quad \mathsf{F}(U) &\dashrightarrow \mathsf{F}(V) \\ f \quad &\dashrightarrow \rho_{U,V}(f) = f|_V. \end{aligned}$$

Therefore, (F, ρ) defines a presheaf of abelian groups on X.

Example 1.2.3. A special case of the above example is when $X = S^1$ with the usual topology induced by R^2 and $A = \mathsf{Z}$. The abelian group is formed again by the continuous functions $f : U \to \mathsf{Z}$. Moreover, when U is connected and non-empty we have $\mathsf{F}(U) \cong \mathsf{Z}$. Indeed, if we take $f \in \mathsf{F}(U)$, then there exists $z \in \mathsf{Z}$ such that $z \in Im(f)$. Since $f^{-1}(z)$ and $f^{-1}(\mathsf{Z} - \{z\})$ is a decomposition of U into disjoint open set, we conclude that $f(x) = z$ for all $x \in U$. Therefore we can identify $\mathsf{F}(U)$ with Z.

Definition 1.2.4. Let (F, ρ) be a presheaf of abelian groups on X. Given any open set $U \subset X$ and an open covering $\{U_i\}_{i\in I}$ of U, we say that (F, ρ) is a *sheaf of abelian groups* on X if it verifies the properties

(i) if $f \in \mathsf{F}(U)$ with $f|_{U_i} = 0$ for all i, then $f = 0$;

(ii) if $f_i \in \mathsf{F}(U_i)$ such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all $i, j \in I$, then there exists $f \in \mathsf{F}(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

Example 1.2.5. The presheaf of abelian groups (F, ρ) given in Example 1.2.2 is a sheaf of abelian groups on X

However, not all presheaves are sheaves.

Example 1.2.6. Take the set $X = \{a, b\}$ with the discrete topology. For each open set $U \subset X$ we define $\mathsf{F}(U)$ as Z if $U = \{a\}$ or $\{b\}$, and $\mathsf{F}(U) = \{0\}$ if $U = X$ or $U = \emptyset$. Let us see why F is a presheaf but not a sheaf. Clearly $\mathsf{F}(U)$ is an abelian group for all $U \subset X$. The restriction maps are defined as $\rho_{X,\{a\}}(0) = 0$, $\rho_{X,\{b\}}(0) = 0$, $\rho_{\{a\},\emptyset}(z) = 0$, and $\rho_{\{b\},\emptyset}(z) = 0$ for all $z \in \mathsf{Z}$. On the other hand, to see that (F, ρ) is not a sheaf, we express $X = \{a\} \cup \{b\}$. Taking $1 \in \mathsf{F}(\{a\}) \cap \mathsf{F}(\{b\})$, where $\{a\} \cap \{b\} = \emptyset$, we have that $\rho_{\{a\},\emptyset}(1) = 0 = \rho_{\{b\},\emptyset}(1)$. However, there is no $z \in \mathsf{X} = \{0\}$ such that $\rho_{X,\{a\}}(z) = 1$.

Example 1.2.7. [The constant sheaf]. Given X and A as in Example 1.2.2, it is easy to see that the presheaf (F, ρ) defined here is a sheaf of abelian groups on X. This sheaf is called *the constant sheaf with values in* A. A more concrete example is given in Example 1.2.3. This constant sheaf is denoted by Z, and its elements as $\mathsf{Z}(U)$ for each open set $U \subset X$.

Given a presheaf F on X, we define the *stalk* of F at a point $x \in X$ as follows. Consider the disjoint union $\bigsqcup_{x\in U} \mathsf{F}(U)$ of all neighbourhood U of x. On this disjoint union we establish the equivalence relation $\sim$ as $f \sim g$, with $f \in \mathsf{F}(U)$ and $g \in \mathsf{F}(V)$, if and only if there exists an open set $W \subset U \cap V$ containing x such that $f|_W = g_W$. The stalk of F in this point x is given by

$$\mathsf{F}_x = \frac{\bigsqcup_{x\in U} \mathsf{F}(U)}{\sim}.$$

It is easy to see that each F_x carries a group structure.

Definition 1.2.8. Let $(\mathsf{F}, \rho), (\mathsf{G}, \overline{\rho})$ be sheaves on X. We define a sheaf homomorphism $\alpha : \mathsf{F} \to \mathsf{G}$ as a set of group homomorphism $\alpha_U : \mathsf{F}(U) \to \mathsf{G}(U)$ (for an open set $U \subset X$) such that if $V \subset U$ is open, then the diagram

$$\begin{tikzcd}
\mathsf{F}(U) \arrow[r, "\alpha_U"] \arrow[d, "\rho_{U,V}"'] & \mathsf{G}(U) \arrow[d, "\overline{\rho}_{U,V}"] \\
\mathsf{F}(V) \arrow[r, "\alpha_V"] & \mathsf{G}(V)
\end{tikzcd}$$

commutes.

In the definition given above, if F and G are presheaves, then α is called a *presheaf homomorphism*.

Proposition 1.2.9. *Let F be a presheaf on X. Then there exists a sheaf $\overline{\mathsf{F}}$ on X and a presheaf homomorphism $i : \mathsf{F} \to \overline{\mathsf{F}}$ such that for any homomorphism $\alpha : \mathsf{F} \to \mathsf{G}$, with G a sheaf, there exists a unique sheaf homomorphism $\overline{\alpha} : \overline{\mathsf{F}} \to \mathsf{G}$ that verifies $\alpha = \overline{\alpha} \circ i$.*

Proof. See Serre [25]. □

The sheaf $\bar{\mathcal{F}}$ obtained above is called a *sheafification* of $\mathcal{F}$.

We define the *kernel* of the sheaf homomorphism $\alpha : \mathcal{F} \to \mathcal{G}$ as the collection $\ker(\alpha) = \{\ker(\alpha)(U)\}$ given by $\ker(\alpha)(U) = \ker(\alpha_U)$ for each open set $U \subset X$. It is clear that $\ker(\alpha)$ establish a sheaf of abelian groups on X.

On the image of a sheaf homomorphism, we notice that the collection $\{Im(\alpha_U)\}$ does not establish a sheaf on X but a presheaf. Then the image of a sheaf homomorphism is defined as the sheafification of the presheaf $\{Im(\alpha_U)\}$.

Let $\{\mathcal{F}^i\}$ be a collection of sheaves on X. A *complex* $\mathcal{F}^\cdot$ of sheaves on X is defined as a sequence of sheaves homomorphisms

$$\cdots \longrightarrow \mathcal{F}^{i-1} \xrightarrow{d^{i-1}} \mathcal{F}^{i} \xrightarrow{d^{i}} \mathcal{F}^{i+1} \xrightarrow{d^{i+1}} \cdots$$

such that $d^i \circ d^{i-1} = 0$ for all $i \in \mathbb{Z}$. The complex $\mathcal{F}^\cdot$ is called *exact* if $Im(d^{i-1}) = \ker(d^i)$.

Let $\mathcal{F}$ be a sheaf on X and $\mathcal{U} = \{U_\lambda\}$ a locally finite cover of open sets on X. We define the set $C^p(\mathcal{U}, \mathcal{F})$ of *p-chains* of $\mathcal{F}$ as

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0,\dots,i_p} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_p}).$$

The elements of $C^p(\mathcal{U}, \mathcal{F})$ are written as $f_{i_0,\dots,i_p}$. We define the group homomorphism δ_p as

$$\begin{aligned} \delta_p : \quad & C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F}) \\ & f_{i_0,\dots,i_p} \longrightarrow \delta_p(f_{i_0,\dots,i_p}) = g_{i_0,\dots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k f_{i_0,\dots,\hat{i}_k,\dots,i_{p+1}} , \end{aligned}$$

where the symbol $\hat{}$ means that we have removed the index below. This homomorphism δ_p is a *coboundary operator*. For instance, when $p = 1$ and $p = 2$ we have

$$\delta_1(f_{i_0,i_1}) = g_{i_0,i_1,i_2} = f_{i_1,i_2} - f_{i_0,i_2} + f_{i_0,i_1} \text{ on } U_{i_0,i_1,i_2},$$
$$\delta_2(f_{i_0,i_1,i_2}) = g_{i_0,i_1,i_2,i_3} = f_{i_1,i_2,i_3} - f_{i_0,i_2,i_3} + f_{i_0,i_1,i_3} - f_{i_0,i_1,i_2} \text{ on } U_{i_0,i_1,i_2,i_3}.$$

Remark 1.2.10. We notice that $\delta_{p+1} \circ \delta_p = 0$ for all $p \in \mathbb{Z}$. This is left as an exercise to the reader. However,we can do this for $p = 1$. Indeed, writing

$g_{i_0,i_1,i_2} = \delta_1(f_{i_0,i_1})$, we have

$$\begin{aligned}\delta_2(g_{i_0,i_1,i_2}) &= g_{i_1,i_2,i_3} - g_{i_0,i_2,i_3} + g_{i_0,i_1,i_3} - g_{i_0,i_1,i_2} \\ &= (f_{i_2,i_3} - f_{i_1,i_3} + f_{i_1,i_2}) - (f_{i_2,i_3} - f_{i_0,i_3} + f_{i_0,i_2}) + \\ &\qquad (f_{i_1,i_3} - f_{i_0,i_3} + f_{i_0,i_1}) - (f_{i_1,i_2} - f_{i_0,i_2} + f_{i_0,i_1}) \\ &= 0.\end{aligned}$$

And thus we get $\delta_2 \circ \delta_1 = 0$.

Given the coboundary operator $\delta_p : C^p(\mathsf{U}, \mathsf{F}) \to C^{p+1}(\mathsf{U}, \mathsf{F})$, we define the subgroups $Z^p(\mathsf{U}, \mathsf{F}) = \ker(\delta_p)$ and $B^p(\mathsf{U}, \mathsf{F}) = Im(\delta_{p-1})$. The elements of $Z^p(\mathsf{U}, \mathsf{F})$ and $B^p(\mathsf{U}, \mathsf{F})$ are called *cocycles* and *coboundaries*, respectively. By above remark we have that $B^p(\mathsf{U}, \mathsf{F}) \subset Z^p(\mathsf{U}, \mathsf{F})$. Next we define the quotient group $H^p(\mathsf{U}, \mathsf{F})$ as

$$H^p(\mathsf{U}, \mathsf{F}) = \frac{Z^p(\mathsf{U}, \mathsf{F})}{B^p(\mathsf{U}, \mathsf{F})}.$$

This group is called the p-th cohomology group with coefficients in F with respect to the covering U.

We notice that $H^p(\mathsf{U}, \mathsf{F})$ depends of the open covering U. To define cohomology groups that depend only of X and F, we should refine the covering and then we take an inductive limit on U (see Serre [25]). This cohomology group is denoted by $H^p(X, \mathsf{F})$.

Consider X to be a paracompact topological space and U a locally finite open covering of X. If we have an exact sequence of sheaves on X like

$$0 \to \mathsf{A} \xrightarrow{\alpha} \mathsf{B} \xrightarrow{\beta} \mathsf{C} \to 0,$$

then it induces an exact sequence in cohomology (see Serre [25])

$$\cdots \to H^p(\mathsf{U}, \mathsf{B}) \to H^p(\mathsf{U}, \mathsf{C}) \to H^{p+1}(\mathsf{U}, \mathsf{A}) \to H^{p+1}(\mathsf{U}, \mathsf{B}) \to \cdots.$$

We have the following result.

Proposition 1.2.11. *Take an exact sequence of cohomology as above. If we take the inductive limit on* U*, the sequence*

$$\cdots \to H^q(X, \mathsf{B}) \to H^q(X, \mathsf{C}) \to H^{q+1}(X, \mathsf{A}) \to H^{q+1}(X, \mathsf{B}) \to \cdots$$

which is exact.

Proof. See Serre [25]. □

1.3 Symplectic manifolds

Consider $\omega : V \times V \to \mathbb{R}$ a skew-symmetric bilinear map. We say that the map ω is *symplectic* when $\omega(u, v) = 0$ for all $v \in V$ implies $u = 0$. We call (V, ω) a *symplectic vector space.*

Proposition 1.3.1. *Let (V, ω) be a symplectic vector space. Then the dimension of V is even. Furthermore there exists a basis $\{e_1, ..., e_n, f_1, ..., f_n\}$ of V such that*

$$\omega(e_i, f_j) = \delta_{ij} \quad \textit{and} \quad \omega(e_i, e_j) = 0 = \omega(f_i, f_j).$$

Proof. See McDuff and Salamon [21]. □

The basis of V mentioned above is called a *symplectic basis.*

Example 1.3.2. On $\mathbb{R}^{2n}$ take the skew-symmetric bilinear map ω_0 defined by

$$\omega_0(x, y) = x_1y_1 + ... + x_ny_n - (x_{n+1}y_{n+1} + ... + x_{2n}y_{2n}),$$

where $x = (x_1, ..., x_{2n})$ and $y = (y_1, ..., y_{2n})$. This map is symplectic. Indeed, if $\omega_0(x, y) = 0$ for all $x, y \in \mathbb{R}^{2n}$, in particular for $y = \tilde{x} = (x_1, ..., x_n, -x_{n+1}, ..., -x_{2n})$ we have $\omega_0(x, \tilde{x}) = x_1^2 + ... + x_n^2 + x_{n+1}^2 + ... + x_{2n}^2 = 0$, which implies $x = 0$. Then $(\mathbb{R}^{2n}, \omega_0)$ is a symplectic vector space. A symplectic basis for this example is given by choosing $\{e_1, ..., e_n, f_1, ..., f_n\}$, where $e_i = (0, ..., 0, \overbrace{1}^{i}, 0, ..., 0)$ and $f_i = (0, ..., 0, \overbrace{1}^{(n+i)}, 0, ..., 0)$.

Let M be a $2n$-dimensional smooth manifold. A 2-form ω on M is called a *symplectic form* if ω is closed ($d\omega = 0$) and ω_p is symplectic on T_pM for each $p \in M$. The pair (M, ω) is a *symplectic manifold.*

Example 1.3.3. On $\mathbb{R}^{2n}$ with linear coordinates $x_1, ..., x_n, y_1, ..., y_n$, we define the 2-form $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$. Clearly ω_0 is a symplectic form. Furthermore, a symplectic basis for $(\mathbb{R}^{2n}, \omega_0)$ is given by taking

$$\{(\frac{\partial}{\partial x_1})_p, ..., (\frac{\partial}{\partial x_n})_p, (\frac{\partial}{\partial y_1})_p, ..., (\frac{\partial}{\partial y_n})_p\}$$

on T_pM .

Example 1.3.4. On S^2, for each $p \in S^2$ we view its tangent space T_pS^2 as the tangent plane to S^2 in p. Clearly T_pS^2 is formed by orthogonal vectors to p in $\mathbb{R}^3$. We define the symplectic form ω on S^2 as $\omega_p(u, v) = (p, u \times v)$. Notice that ω is closed. To show that ω_p is symplectic, we notice that $(\omega_p)_u(v) = \omega_p(u, v)$ is 0 if and only if $(p, u \times v) = 0$. Thus $(\omega_p)_u(v) = 0$ whenever $u \times v = 0$. Then $\omega_p(u, v) = 0$ for all $v \in T_pS^2$ is equivalent to $u = 0$.

Remark 1.3.5. Although S^2 has a symplectic structure, no other sphere S^{2n}, with $n > 1$, admits a symplectic structure. See McDuff and Salamon [21].

1.4 Complex structures on manifolds

A *complex manifold* of complex dimension n is a topological manifold M together with an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$, where $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$, in which the maps $\varphi_\alpha \circ \varphi_\beta^{-1}$ (called *transition functions*) are holomorphic on $\varphi_\beta(U_\alpha \cap U_\beta)$. This last condition is called *compatibility condition.* The atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ is called a *holomorphic structure* on M .

We notice that a complex manifold of dimension n defines a real smooth manifold of dimension $2n$, since we can identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and the transition functions with smooth maps between open subsets of $\mathbb{R}^{2n}$. The converse is not necessarily true, because not every smooth map is holomorphic.

Example 1.4.1. [**The Riemann sphere S^2.**] Let S^2 be the unit sphere in $\mathbb{R}^3$. An open covering for S^2 is given by the sets $U_1 = S^2 \setminus \{(0, 0, 1)\}$ and $U_2 = S^2 \setminus \{(0, 0, -1)\}$. On U_1 we define the map $\varphi_1 : U_1 \to \mathbb{C}$ by $\varphi(a, b, c) = \frac{a}{1-c} + i\frac{b}{1-c}$. For U_2, the map $\varphi_2 : U_2 \to \mathbb{C}$ is defined by $\varphi_2(a, b, c) = \frac{a}{1+c} - i\frac{b}{1+c}$. Therefore, for $z \in \mathbb{C} \setminus \{0\}$ we have $\varphi_1 \circ \varphi_2^{-1}(z) = z^{-1}$. Thus, the transition functions are holomorphic, and S^2 is a complex manifold of complex dimension 1.

Example 1.4.2. [**The complex projective space $\mathbb{CP}^n$.**] On the set $\mathbb{C}^{n+1} \setminus \{0\}$, we impose the equivalence relation $\sim$ as

$$(z_0, ..., z_n) \sim \lambda(z_0, ..., z_n) \text{ for all } \lambda \in \mathbb{C}^*.$$

We denote the equivalence class of $(z_0, ..., z_n)$ by $[z_0 : \cdots : z_n]$, and the quotient space by $\mathbb{CP}^n$. An open covering for $\mathbb{CP}^n$ is given by the collection of open charts $\{U_k\}_{k=0}^n$, where $U_k = \{[z_0 : ... : z_n] / z_k \neq 0\}$. On each U_k, we define φ_k by

$$\varphi_k([z_0 : ... : z_n]) = \left(\frac{z_0}{z_k}, ..., \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, ..., \frac{z_n}{z_k}\right).$$

The transition functions $\varphi_k \circ \varphi_j^{-1}$ on $\varphi_j(U_k \cap U_j)$ are obtained as

$$\begin{aligned}\varphi_k \circ \varphi_j^{-1}(w_1, ..., w_n) &= \varphi_k([w_1 : ... w_j : 1 : w_{j+1} : ... w_n]) \\ &= \left(\frac{w_1}{w}, ..., \frac{w_j}{w}, \frac{1}{w}, \frac{w_{j+1}}{w}, ..., \frac{w_{k-1}}{w}, \frac{w_{k+1}}{w}, ..., \frac{w_n}{w}\right).\end{aligned}$$

Clearly, the transition functions are holomorphic on their domains. Then $\mathbb{C}P^n$ is a complex manifold of dimension n.

An *almost complex structure* on a real vector space V is an endomorphism J such that $J^2 = -Id$. We denote it by (V, J). Observe that V necessarily has even dimension, this because of the relation $(\det(J))^2 = (-1)^n$, where n is the dimension of V; it follows that n is necessarily even.

Definition 1.4.3. An *almost complex structure* on a smooth manifold M is a smooth endomorphism field $J : TM \to TM$ such that (T_xM, J_x) is a vector space with an almost complex structure for all $x \in M$. We denote it by (M, J) and called it an *almost complex manifold.*

By the above remark, T_xM must have even real dimension, so M has even real dimension as well.

Example 1.4.4. [**The 2-sphere**.] The sphere S^2 is an almost complex manifold. In fact, we define on each fiber T_uS^2 the endomorphism $J_u : T_uS^2 \to T_uS^2$ by $J_u(v) = u \times v$, where $\times$ is the cross product in $\mathbb{R}^3$. Then we have $J_u^2(v) = u \times (u \times v) = -v$. For a point $P(x, y, z)$ in S^2, we choose the basis $\frac{\partial}{\partial x} = (1 - z - x^2, -yx, x - zx)$ and $\frac{\partial}{\partial y} = (-xy, 1 - z - y^2, y - zy)$. Notice the relation $J\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x}$. See Figure 1.1.

Example 1.4.5. [**The 6-sphere.**] This example is taken form Simanca [26]. To define an almost complex structure on S^6, we use the cross product on $\mathbb{R}^7$. Remember that if u_1, u_2 and u_3 are orthogonal unit vectors on $\mathbb{R}^7$, then the set

$$\{u_1, u_2, u_1 \times u_2, u_3, u_1 \times u_3, u_2 \times u_3, (u_1 \times u_2) \times u_3\}$$

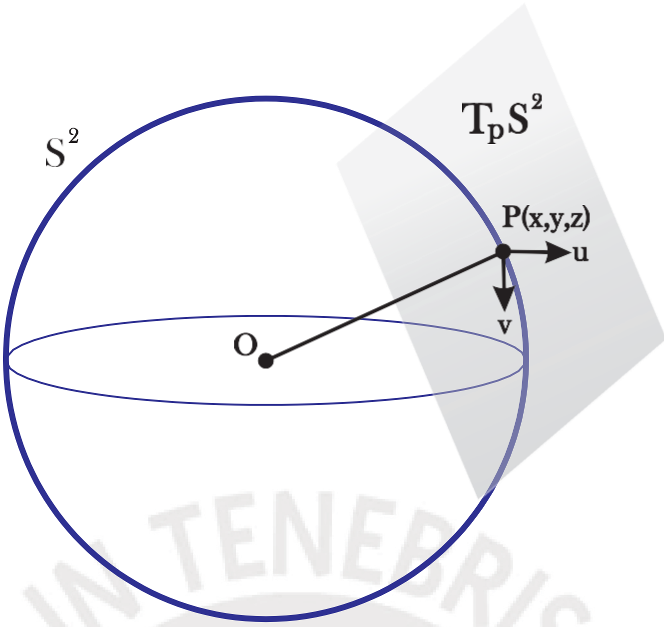

Figure 1.1: Taking $u = \frac{\partial}{\partial x}$ and $v = \frac{\partial}{\partial y}$ as in the Example 1.4.4, we obtain $J\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x}$.

is an orthonormal set (with respect to dot product). For sake of simplicity we take the following notation: $e_1 = u_1; e_2 = u_2; e_3 = u_1 \times u_2; e_4 = u_3;$ $e_5 = u_1 \times u_3$; $e_6 = u_2 \times u_3$; $e_7 = (u_1 \times u_2) \times u_3$. Then, using the properties of cross product, we arrive to the following "multiplication table"

$\times$	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	0	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	0	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	0	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	0	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	0	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	0	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	0

Computing $u \times v$, we obtain the equality $u \times v = \sum_{i,j=1}^{7} u_i v_j e_i \times e_j$. We define on each fiber T_uS^6, the endomorphism $J_u : T_uS^6 \to T_uS^6$ as $J_u(v) = u \times v$.

Observe that J_u is well defined, because of $u \cdot (u \times v) = 0$.

To prove that $J_u^2 = -Id$ holds for all $u \in S^6$, we use the property of cross product

$$u \times (v \times w) = (u \cdot w)v - (u \cdot v)w.$$

Thus, we obtain

$$J_u^2(v) = u \times (u \times v) = (u \cdot v)u - (u \cdot u)v = -v.$$

Next we explore the link between almost complex manifolds and complex manifolds.

If we have a complex manifold M^n, we can easily define an almost complex structure on M. First we identify M with a $2n$-dimensional real smooth manifold, and a complex chart $(U_\alpha, \varphi_\alpha)$ on M with a real chart. We define on each fiber T_xM, where $x \in U_\alpha$, the linear map $J_\alpha = (\varphi_\alpha)_*^{-1} \circ J_n \circ (\varphi_\alpha)_*$, where

$$J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

It is easy to check that we get $J_\alpha^2 = -Id$ on each T_xM. Then we define the endomorphism J on TM as $J(u) = J_\alpha(u)$ if $u \in T_xM$, where $x \in U_\alpha$. Let us see that J is well-defined. Taking another chart (U_β, φ_β), where $x \in U_\beta$, we have another linear map J_β defined on T_xM by $J_\beta = (\varphi_\beta)_*^{-1} \circ J_m \circ (\varphi_\beta)_*$. We ask if J_α and J_β agree on T_xM. To settle this question, for v an element of T_xM we have

$$\begin{aligned} J_\beta(v) &= (\varphi_\beta)_*^{-1} \circ J_n \circ (\varphi_\beta)_*(v) \\ &= (\varphi_\beta)_*^{-1} \circ J_n \circ (\varphi_\beta)_* \circ (\varphi_\alpha)_*^{-1} \circ (\varphi_\alpha)_*(v). \end{aligned}$$

Since $(\varphi_\beta)_* \circ (\varphi_\alpha)_*^{-1}$ is holomorphic, we have that $(\varphi_\beta)_* \circ (\varphi_\alpha)_*^{-1}$ and J_n commute. Thus, we obtain

$$\begin{aligned} J_\beta(v) &= (\varphi_\beta)_*^{-1} \circ (\varphi_\beta)_* \circ (\varphi_\alpha)_*^{-1} \circ J_n \circ (\varphi_\alpha)_*(v) \\ &= (\varphi_\alpha)_*^{-1} \circ J_n \circ (\varphi_\alpha)_*(v) \\ &= J_\alpha(v). \end{aligned}$$

Then J is well-defined and clearly verifies $J^2 = -Id$.

We must stress the fact that not all almost complex structures arise from a holomorphic structure. For example this happens in S^6 (see Simanca [26]). The Newlander-Nirenberg theorem provides a criterion to settle this question. Before stating the theorem, we give some definitions and results regarding complex linear algebra.

Let (V, J) be a vector space with an almost complex structure. Then on the complexification $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ of V, we can extend J to a complex endomorphism $J_{\mathbb{C}}$ as

$$J_{\mathbb{C}}(v \otimes \alpha) = J(v) \otimes \alpha.$$

By simplicity we still denote $J_{\mathbb{C}}$ as J. Notice that the eigenvalues of J are $i, -i$, which naturally determine two eigenspaces

$$V^{1,0} = \{w \in V_{\mathbb{C}} : Jw = iw\} = \{v - iJv : v \in V\},$$
$$V^{0,1} = \{w \in V_{\mathbb{C}} : Jw = -iw\} = \{v + iJv : v \in V\}.$$

Now, consider V^*, be the dual space of V. This V^* admits an almost complex structure arising from J, too. In fact, we define this structure $J^* : V^* \to V^*$ by $J^*(v^*)(u) = v^*(Ju)$, where $v^* \in V^*$, $u \in V$. Since $J^2 = -Id$, it is easy to verify that we get

$$(J^*)^2(v^*)(u) = (J^*(v^*(Ju)) = v^*(J^2(u)) = v^*(-u) = -v^*(u).$$

So, being V^* also a vector space with almost complex structure, we can, analogously, extend J^* to a complex endomorphism $J^*_{\mathbb{C}}$ on $V^*_{\mathbb{C}} = V^* \otimes \mathbb{C}$. We will still denote J^* and $J^*_{\mathbb{C}}$ as J. Clearly this endomorphism $J : V^*_{\mathbb{C}} \to V^*_{\mathbb{C}}$ has two eigenvalues i and $-i$ with eigenspaces

$$V_{1,0} = \{\xi \in V_{\mathbb{C}}{}^* : J\xi = i\xi\},$$

$$V_{0,1} = \{\xi \in V_{\mathbb{C}}{}^* : J\xi = -i\xi\},$$

respectively.

Proposition 1.4.6. *Let $V_{\mathbb{C}}^*$ be the complexification of the dual space V^* of V. If $V_{1,0}$ and $V_{0,1}$ are as above, then*

$$V_{1,0} = \{\xi \in V_{\mathbb{C}}{}^* : \xi(v) = 0 \quad \textit{for all } v \in V^{0,1}\},$$

$$V_{0,1} = \{\xi \in V_{\mathbb{C}}{}^* : \xi(v) = 0 \quad \textit{for all } v \in V^{1,0}\}.$$

Proof. Let $\xi \in V_{1,0}$. Since $J\xi = i\xi$, we have $\xi \circ J(w) = i\xi(w)$ for all $w \in V_{\mathbb{C}}$. Expressing w as $w = v^{1,0} + v^{0,1} \in V^{1,0} + V^{0,1}$, we have $\xi \in V_{1,0}$ if and only if $\xi \circ J(v^{1,0} + v^{0,1}) = i\xi(v^{1,0} + v^{0,1}) = i\xi(v^{1,0}) + i\xi(v^{0,1})$. Thus, since $\xi \circ J(v^{1,0} + v^{0,1}) = i\xi(v^{1,0}) - i\xi(v^{0,1})$, we conclude that $\xi \in V_{1,0}$ if and only if $\xi(v) = 0$ for all $v \in V^{0,1}$. For $V^{0,1}$ the proof is similar. □

Let (M^{2m}, J) be an almost complex manifold. By the above, the complexification $TM^{\mathbb{C}} = TM \otimes_{\mathbb{R}} \mathbb{C}$ has an almost complex structure J. The endomorphism J splits the complexified tangent bundle into bundles of eigenspaces $TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$, where

$$T^{1,0}M = \{Z \in TM^{\mathbb{C}} : J(Z) = iZ\} = \{X - iJX : X \in TM\},$$
$$T^{0,1}M = \{Z \in TM^{\mathbb{C}} : J(Z) = -iZ\} = \{X + iJX : X \in TM\}.$$

In what follows, we will find a basis for $T^{0,1}M$ and $T^{1,0}M$ in the case when J arises from a holomorphic structure.

If this is the case, we can express locally J as

$$J_a = (\varphi_a)_*^{-1} \circ J_m \circ (\varphi_a)_*,$$

where (U_a, φ_a) is a holomorphic chart on M. Writing $\varphi_a = (z_1, ..., z_m)$ gives us real coordinates $(x_1, ..., x_m, y_1, ..., y_m)$ on U_a, where $z_k = x_k + iy_k$. Thus, we obtain a local frame $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m}, \frac{\partial}{\partial y_1}, ..., \frac{\partial}{\partial y_m}\}$, with $\frac{\partial}{\partial x_k} = (\varphi_a)_*^{-1}(e_k)$ and $\frac{\partial}{\partial y_k} = (\varphi_a)_*^{-1}(e_{m+k})$, where $\{e_1, ..., e_{2m}\}$ stands for the canonical basis of $\mathbb{R}^{2m}$. Since $J_m(e_k) = e_{k+m}$, we obtain

$$\begin{aligned}
J\left(\frac{\partial}{\partial x_k}\right) &= (\varphi_a)_*^{-1} \circ j_m \circ (\varphi_a)_*\left(\frac{\partial}{\partial x_k}\right) \\
&= (\varphi_a)_*^{-1} \circ j_m(e_k) \\
&= (\varphi_a)_*^{-1}(e_{m+k}) \\
&= \frac{\partial}{\partial y_k}.
\end{aligned}$$

We know that any element $Z \in T^{1,0}M$ splits as $Z = X - iJX$, where $X \in TM$. Expressing X as $X = a_k\frac{\partial}{\partial x_k} + b_k\frac{\partial}{\partial y_k}$, we have them

$$\begin{aligned}
Z &= a_k\frac{\partial}{\partial x_k} + b_k\frac{\partial}{\partial y_k} - iJ\left(a_k\frac{\partial}{\partial x_k} + b_k\frac{\partial}{\partial y_k}\right) \\
&= a_k\frac{\partial}{\partial x_k} + b_kJ\frac{\partial}{\partial x_k} - ia_kJ\frac{\partial}{\partial x_k} + ib_k\frac{\partial}{\partial x_k} \\
&= (a_k + ib_k)\left(\frac{\partial}{\partial x_k} - iJ\frac{\partial}{\partial x_k}\right).
\end{aligned}$$

Thus, we obtain a basis for $T^{1,0}M$ given by

$$\frac{\partial}{\partial z_k} = \frac{1}{2}(\frac{\partial}{\partial x_k} - iJ\frac{\partial}{\partial x_k}) = \frac{1}{2}(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}).$$

Analogously, a basis for $T^{0,1}M$ is given by

$$\frac{\partial}{\partial \overline{z_k}} = \frac{1}{2}(\frac{\partial}{\partial x_k} + iJ\frac{\partial}{\partial x_k}) = \frac{1}{2}(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}).$$

Let M be a smooth manifold of real dimension n. A *r-dimensional distribution* D on M consists in an assignment of vector subspaces $\mathsf{D}_x \subset T_xM$ of dimension r. Furthermore, if for each $x \in M$ there exists a submanifold N containing x such that $T_yN = \mathsf{D}_y$ for all $y \in N$, then the distribution D is called *completely integrable.*

A distribution D is said *involutive* whenever for any $X, Y \in \mathsf{D}$, $[X, Y] \in \mathsf{D}$ holds. A celebrated theorem of Frobenius says that a distribution D is completely integrable if and only if D is involutive (see Morita [22]).

Using this theorem of Frobenius, we have the following result to characterize a complex structure.

Theorem 1.4.7 (The Newlander - Nirenberg theorem)**.** *Let* (M, J) *be an almost complex manifold. The almost complex structure* J *arises from a holomorphic structure when the distribution* $T^{0,1}M$ *is integrable.*

Proof. See Newlander and Nirenberg [24]. □

When the almost complex structure J of M arises from a holomorphic structure, we call J a *complex structure.*

Consider (M^{2m}, J) an almost complex manifold. Let $\Lambda^1 M$ be the exterior bundle of M. Writing $\Lambda^1_{\mathbb{C}} M = \Lambda^1 M \otimes \mathbb{C}$ for the complexified of $\Lambda^1 M$, it splits as

$$\Lambda^1_{\mathbb{C}} M = \Lambda^{1,0}M \oplus \Lambda^{0,1}M,$$

where

$$\Lambda^{1,0}M = \{\xi \in \Lambda^1_{\mathbb{C}} M : J\xi = i\xi\},$$

$$\Lambda^{0,1}M = \{\xi \in \Lambda^1_{\mathbb{C}} M : J\xi = -i\xi\}.$$

By Proposition 1.4.6, we can also express $\Lambda^{1,0}M$ and $\Lambda^{0,1}M$ as

$$\Lambda^{1,0} = \{\xi \in \Lambda^1_{\mathbb{C}} M : \xi(Z) = 0 \quad \text{for all } Z \in T^{0,1}M\},$$
$$\Lambda^{0,1} = \{\xi \in \Lambda^1_{\mathbb{C}} M : \xi(Z) = 0 \quad \text{for all } Z \in T^{1,0}M\},$$

respectively. Let $\Omega^{1,0}M$ be the space of sections on $\Lambda^{1,0}M$. The elements of $\Omega^{1,0}M$ are called (1, 0)-forms. In the same way we can define $\Omega^{0,1}M$.

If J is a complex structure on M , then for a holomorphic chart (U_a, φ_a) on M , we can express φ_a in local coordinates as $\varphi_a = (z_1, ..., z_m)$. Moreover, each z_k can be written as $z_k = x_k + iy_k$. Thus, we obtain real coordinates $(x_1, ..., x_m, y_1, ..., y_m)$ on U_a. We define the complex 1-forms

$$dz_k = dx_k + idy_k, \qquad d\bar{z}_k = dx_k - iy_k.$$

The set $\{dz_1, ..., dz_m\}$ forms a local basis for $\Lambda^{1,0}M$. It is easy to see that this set is linearly independent. Let us check that each element dz_k belongs to $\Lambda^{1,0}M$. For that, we take $W \in T^{0,1}M$ and prove that $dz_k(W) = 0$ holds. Let $W = \sum_j \alpha_j \frac{\partial}{\partial \bar{z}_j}$ be in $T^{0,1}M$, where $\alpha_j = a_j + ib_j$. Then we have

$$\begin{aligned} dz_k(W) &= (dx_k + idy_k)\left(\sum_j (a_j + ib_j)\frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y_j}\right)\right) \\ &= \frac{1}{2}(a_k + ib_k + i(ia_k - b_k)) \\ &= 0. \end{aligned}$$

Because of $\Lambda^{1,0}M$ has dimension m, the collection $\{dz_1, ..., dz_m\}$ is a basis for $\Lambda^{1,0}M$. Furthermore, this basis is the associated dual basis to $\{\frac{\partial}{\partial z_1}, ..., \frac{\partial}{\partial z_m}\}$. In a similar fashion $\{d\bar{z}_1, ..., d\bar{z}_m\}$ is a basis for $\Lambda^{0,1}M$, the associated dual basis to $\frac{\partial}{\partial \bar{z}_1}, ..., \frac{\partial}{\partial \bar{z}_m}$.

Let $\Lambda^k_{\mathbb{C}} M$ be the complexification of $\Lambda^k M$ the k-exterior bundle of M . Denote by $\Lambda^{p,0}$ the pth exterior power of $\Lambda^{1,0}M$, and by $\Lambda^{0,q}$ the qth exterior power of $\Lambda^{0,1}M$. Using the fact that the exterior power of a direct sum of vector spaces satisfies

$$\Lambda^k(E \oplus F) \cong \bigoplus_{j=0}^{k}(\Lambda^j E \otimes \Lambda^{k-j} F),$$

we have, for $\Lambda^k_{\mathbb{C}} M$, the equality

$$\Lambda^k_{\mathbb{C}} M \cong \bigoplus_{p+q=m} (\Lambda^{p,q} M),$$

where $\Lambda^{p,q}M = \Lambda^{p,0}M \otimes \Lambda^{0,q}M$.

The space of sections on $\Lambda^{p,q}M$ is denoted by $\Omega^{p,q}M$, and its elements are called (p, q)-*forms*. A basis for $\Lambda^{p,q}M$ is given then by

$$\{dz_{j_1} \wedge ... \wedge dz_{j_p} \wedge d\bar{z}_{k_1} \wedge ... \wedge d\bar{z}_{k_q} : j_1 < ... < j_p, k_1 < ... < k_q\}.$$

Recall that an almost complex structure J arises from a holomorphic structure on M as long as the distribution $T^{0,1}M$ is integrable. In practice, this means that if we take two elements $X + iJX$ and $Y + iJY$ in $T^{0,1}M$, where $X, Y \in TM$, then we must have $[X + iJX, Y + iJY] \in T^{0,1}M$. Calculating $[X + iJX, Y + iJY]$, we get

$$\begin{aligned}[X + iJX, Y + iJY] &= [X, Y] + i[X, JY] + i[JX, Y] - [JX, JY]\\ &= [X, Y] - [JX, JY] + i([X, JY] + [JX, Y]).\end{aligned}$$

Supposing that $T^{0,1}M$ is integrable, $[X + iJX, Y + iJY]$ is of the form $\eta + iJ\eta$, with $\eta \in TM$. By the above equality, we obtain

$$[X, JY] + [JX, Y] = J([X, Y] - [JX, JY]).$$

Since this last equality is equivalent to

$$J([X, JY] + [JX, Y]) = -[X, Y] + [JX, JY],$$

we can conclude that $T^{0,1}M$ is integrable as long as

$$[JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] = 0.$$

The above calculations motivate us to define the *Nijenhuis tensor* N^J as

$$N^J(X, Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y].$$

In brief, J is a complex structure provided that $N^J \equiv 0$ holds. Notice that $N^J \equiv 0$ means that the distribution $T^{1,0}$ is integrable. Indeed, taking $X - iJX, Y - iJY \in T^{1,0}M$, we have $[X - iJX, Y - iJY] \in$

$T^{1,0}M$ if and only if $[X, Y] - [JX, JY] - i([X, JY] + [JX, Y]) \in T^{1,0}M$. Since the elements in $T^{1,0}M$ can be expressed as $\eta - iJ\eta$, we obtain $[X, JY] + [JX, Y] = J([X, Y] - [JX, JY])$. This last equality is equivalent to $N^J(X, Y) = 0$.

The next proposition presents other necessary and sufficient conditions in order that J can be turned a complex structure on M. The proof is taken from Moroianu [23].

Proposition 1.4.8. *The following statements are equivalent:*

(1) $N^J \equiv 0$*,*

(2) $d(\Omega^{1,0}M) \subset \Omega^{2,0}M \oplus \Omega^{1,1}M$*,*

(3) $d(\Omega^{p,q}M) \subset \Omega^{p+1,q}M \oplus \Omega^{p,q+1}M$*.*

Proof. First, we prove that (2) is equivalent to (1). Let ξ be in $\Omega^{1,0}M$. We write $d\xi$ as

$$d\xi = \xi^{2,0} + \xi^{1,1} + \xi^{0,2} \in \Omega^{2,0}M \oplus \Omega^{1,1}M \oplus \Omega^{0,2}M.$$

Then we have $d(\Omega^{1,0}M) \subset \Omega^{2,0}M \oplus \Omega^{1,1}M$ if and only if $\xi^{0,2} \equiv 0$. This last equality happens provided we get $\xi^{0,2}(W, Z) = 0$, for all W, Z in $T^{0,1}M$. From $\xi^{0,2} = d\xi - \xi^{2,0} - \xi^{1,1}$ we obtain

$$\xi^{0,2}(W, Z) = d\xi(W, Z) - \xi^{2,0}(W, Z) - \xi^{1,1}(W, Z) = 0$$

if and only if $d\xi(W, Z) = 0$; here $W, Z \in T^{0,1}M$. This last equality is equivalent to $\frac{1}{2}\{W(\xi(Z)) - Z(\xi(W)) - \xi([W, Z])\} = 0$. As $\xi \in \Omega^{1,0}M$, we obtain $\xi(Z) = \xi(W) = 0$. Thus we get $\xi([W, Z]) = 0$ and $[W, Z] \in T^{0,1}M$, which allows us to conclude the equality $N^J \equiv 0$.

Reciprocally, if we assume $N^J \equiv 0$, we obtain

$$d(\Omega^{0,1}M) \subset \Omega^{1,1}M \oplus \Omega^{0,2}M.$$

Indeed, taking $\eta \in \Omega^{0,1}M$, we have

$$d\eta = \eta^{2,0} + \eta^{1,1} + \eta^{0,2} \in \Omega^{2,0}M \oplus \Omega^{1,1}M \oplus \Omega^{0,2}M.$$

Then $\eta^{2,0}(W, Z) = 0$ if and only if $d\eta(W, Z) - \eta^{1,1}(W, Z) - \eta^{0,2}(W, Z) = 0$, for $W, Z \in T^{1,0}M$. As $\eta^{1,1}(W, Z) = 0$ and $\eta^{0,2}(W, Z) = 0$ for any W, Z in $T^{1,0}M$, we get $\eta^{2,0}(W, Z) = 0$ if and only if $d\eta(W, Z) = 0$.

Finally, because of $d\eta(W, Z) = \frac{1}{2}\{W(\eta(Z)) - Z(\eta(W)) - \eta([W, Z])\}$ and $\eta^{0,1} \in \Omega^{0,1}M$, we obtain that $\eta^{2,0}(W, Z) = 0$ is equivalent to $\eta([W, Z]) = 0$. This last equivalence means $[W, Z] \in T^{1,0}M$. As a consequence $T^{1,0}M$ is integrable.

Next, let us see that (2) implies (3). Taking $\omega \in \Omega^{p,q}M$, we can express ω as

$$\omega = \xi^1 \wedge \dots \wedge \xi^p \wedge \eta^1 \wedge \dots \wedge \eta^q,$$

where $\xi^j \in \Omega^{1,0}M$ and $\eta^k \in \Omega^{0,1}M$. Then we have

$$d\omega = \sum_{j=1} (-1)^{j+1}\xi^1 \wedge \dots \wedge d\xi^j \wedge \dots \wedge \xi^p \wedge \eta^1 \wedge \dots \wedge \eta^q + \sum_{k=1} (-1)^{k+p+1}\xi^1 \wedge \dots \wedge \xi^p \wedge \eta^1 \wedge \dots \wedge d\eta^k \wedge \dots \wedge \eta^q.$$

As (2) implies (1), this results in $d\xi^j \in \Omega^{2,0}M \oplus \Omega^{1,1}M$ and $d\eta^k \in \Omega^{1,1}M \oplus \Omega^{0,2}M$. Therefore, we have

$$\sum_{j=1} (-1)^{j+1}\xi^1 \wedge \dots \wedge d\xi^j \wedge \dots \wedge \xi^p \wedge \eta^1 \wedge \dots \wedge \eta^q \in \Omega^{p+1,q}M \oplus \Omega^{p,q+1}M,$$

and

$$\sum_{k=1} (-1)^{k+p+1}\xi^1 \wedge \dots \wedge \xi^p \wedge \eta^1 \wedge \dots \wedge d\eta^k \wedge \dots \wedge \eta^q \in \Omega^{p+1,q}M \oplus \Omega^{p,q+1}M.$$

Hence we get $d\omega \in \Omega^{p+1,q}M \oplus \Omega^{p,q+1}M$.

Finally, taking $p = 1$ and $q = 0$ in (3), we obtain (2). □

Let (M^{2m}, J) be a complex manifold. We recall that a smooth map $f : M \to \mathbb{C}$ is called *holomorphic* if $f \circ \varphi_a^{-1} : \varphi_a(U_a) \to \mathbb{C}$ is a holomorphic map for all complex charts (U_a, φ_a).

We can also characterize holomorphic maps by intrinsic means.

Proposition 1.4.9. *Let $f : M \to \mathbb{C}$ be a smooth map. The following statements are equivalent:*

(1) f is holomorphic,

(2) $Z(f) = 0$*, for all* $Z \in T^{0,1}M$*,*

(3) df *is a* $(1,0)$*-form.*

Proof. It is easy to see that (2) is the same as (3), as $df \in \Omega^{1,0}M$ is equivalent to $Z(f) = df(Z) = 0$ for all $Z \in T^{1,0}M$.

On the other hand, by definition f is holomorphic as long as the map $f \circ \varphi_a^{-1} : \varphi_a(U_a) \to \mathbb{C}$ is holomorphic for any complex chart (U_a, φ_a). Before showing that (1) is equivalent to (3), we remember that a map $g : U \subset \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic provided it satisfies

$$g_* \circ J_n = J_m \circ g_*,$$

where g_* represents the derivative of g as a smooth map, and J_k stands for the standard matrix

$$\begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}.$$

In this context, we have on each chart (U_a, φ_a) that f is holomorphic if and only if satisfies $(f \circ \varphi_a^{-1})_* \circ J_m = J_1 \circ (f \circ \varphi_a^{-1})_*$.

Then, supposing that f is holomorphic, we obtain

$$\begin{aligned} J_1 \circ f_* &= J_1 \circ (f_* \circ (\varphi_a)_*^{-1}) \circ (\varphi_a)_* \\ &= (f \circ \varphi_a^{-1})_* \circ J_m \circ (\varphi_a)_* \\ &= f_* \circ (\varphi_a)_*^{-1} \circ J_m(\varphi_a)_* \\ &= f_* \circ J, \end{aligned}$$

where J is the complex structure on M. Then taking a smooth vector field X on M, we have

$$df(JX) = f_*(JX) = J_1 \circ f_*(X) = idf(X),$$

So we get $df(JX - iX) = 0$. But his implies $df(X + iJX) = 0$ so we have $df \in \Omega^{1,0}M$.

For the converse, if df is a $(1,0)$-form, then we have $df(X + iJX) = 0$ for all $X \in TM$. It implies $df(X) = -idf(JX)$. Multiplying by i both sides we obtain $df(JX) = idf(X)$. This is equivalent to $J_1 \circ f_* = f_* \circ J$. Thus, f is holomorphic. □

By Proposition 1.4.8, we have

$$d(\Omega^{p,q}M) \subset \Omega^{p+1,q}M \oplus \Omega^{p,q+1}M,$$

and this allows us to define two differential operators ∂ and $\bar{\partial}$ on $\Omega^{p,q}M$. For each $\omega \in \Omega^{p,q}M$ we take $d\omega = \omega' + \omega''$, where $\omega' \in \Omega^{p+1,q}M$ and $\omega'' \in \Omega^{p,q+1}M$. Then we write $\partial\omega = \omega'$ and $\bar{\partial}\omega = \omega''$. Clearly, by construction, we have $d = \partial + \bar{\partial}$.

Proposition 1.4.10. *We have the following identities*

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

Proof. Let ω be an element of $\Omega^{p,q}M$. Then we always have $d^2\omega = 0$. Thus, since $d = \partial + \bar{\partial}$, we have $d^2 = \partial^2 + \bar{\partial}^2 + \partial\bar{\partial} + \bar{\partial}\partial$. As $\partial^2\omega \in \Omega^{p+2,q}M$, $\bar{\partial}^2\omega \in \Omega^{p,q+2}M$ and $(\partial\bar{\partial} + \bar{\partial}\partial)\omega \in \Omega^{p+1,q+1}M$, we conclude the equalities $\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$. □

Lemma 1.4.11 (Dolbeault's lemma)**.** *A $\bar{\partial}$-closed* (0, 1)*-form is locally $\bar{\partial}$-exact.*

Proof. See Griffiths and Harris [13]. □

Finally, we can state and prove the following proposition. The proof is taken from Moroianu [23].

Proposition 1.4.12 (The local $i\partial\bar{\partial}$-lemma)**.** *Let ω be a real 2-form of type* $(1,1)$ *on M. Then ω is closed if and only if for any point $x \in M$ there exists a neighbourhood U containing x such that the form ω on U is equal to $i\partial\bar{\partial}u$ for some real function u on U.*

Proof. Suppose $\omega = i\partial\bar{\partial}u$ for some real function u on U. Then thanks to Proposition 1.4.10 we have

$$d\omega = (\partial + \bar{\partial})\omega = (\partial + \bar{\partial})(i\partial\bar{\partial}u) = i\partial^2\bar{\partial}u - i\bar{\partial}^2\partial u = 0.$$

Conversely, suppose ω is a closed real 2-form of type (1,1). By Poincaré lemma, there exists locally a real 1-form σ such that $d\sigma = \omega$. Then we can decompose σ as

$$\sigma = \sigma^{1,0} + \sigma^{0,1} \in \Omega^{1,0}M \oplus \Omega^{0,1}M.$$

Thus, we have

$$\omega = d\sigma = d\sigma^{1,0} + d\sigma^{0,1} = \partial\sigma^{1,0} + \overline{\partial}\sigma^{1,0} + \partial\sigma^{0,1} + \overline{\partial}\sigma^{0,1}.$$

As ω is of type (1,1), we get $\partial\sigma^{1,0} = 0$, $\overline{\partial}^{0,1} = 0$ and $\omega = \overline{\partial}\sigma^{1,0} + \partial\sigma^{0,1}$. Since we have $\overline{\partial}\sigma^{0,1} = 0$, Dolbeault's lemma yields $\sigma^{0,1} = \overline{\partial} f$ for some local function f. Furthermore, because $\sigma^{1,0} = \overline{\sigma^{0,1}}$, we obtain $\sigma^{1,0} = \overline{\overline{\partial} f} = \partial\overline{f}$. Then we get

$$\omega = \overline{\partial}\sigma^{1,0} + \partial\sigma^{0,1} = \overline{\partial}\partial\overline{f} + \partial\overline{\partial} f = i\partial\overline{\partial}(2Im(f)).$$

With $u = 2Im(f)$ we settle the proposition. □

1.5 Kähler structures

Let (M, J) be a complex manifold and g be a Riemannian metric on M. We say that g is a *Hermitian metric* on M if

$$g(JX, JY) = g(X, Y) \quad \text{for all } X, Y \in TM.$$

It is also said that g is *compatible* with J.

In addition, we can define the 2-form ω_g by

$$\omega_g(X, Y) = g(JX, Y).$$

This 2-form is called the *associated Kähler form* to g.

We say that g is a *Kähler metric* and (M, g, J) is a *Kähler manifold* if the 2-form ω_g is closed (that is, $d\omega_g = 0$).

Using the metric g and the 2-form ω_g, we define the linear form h on TM as

$$h(X, Y) = g(X, Y) - i\omega_g(X, Y).$$

Proposition 1.5.1. *The linear form h defined above is a positive hermitian form on TM.*

Proof. Since g and ω_g are $\mathbb{R}$-linear, h is $\mathbb{R}$-linear. On the other hand, if we take $X \neq 0$ in TM, then we have

$$h(X, X) = g(X, X) - i\omega_g(X, X).$$

Due to that g is compatible with J, we get

$$g(JX, X) = g(J^2X, JX) = g(-X, X) = -g(JX, X).$$

This implies that $\omega_g(X, X) = g(JX, X) = 0$. As a consequence, we have $h(X, X) = g(X, X) > 0$ for all $X \neq 0$.

Now we only need to verify that $h(X, Y) = \overline{h(Y, X)}$ holds. By definition, we have $h(Y, X) = g(Y, X) - i\omega_g(Y, X)$. Taking its conjugate, we arrive to

$$\overline{h(Y, X)} = g(Y, X) + i\omega_g(Y, X).$$

Finally, as $\omega_g(Y, X) = -\omega_g(X, Y)$, we obtain

$$\overline{h(Y, X)} = g(X, Y) - i\omega_g(X, Y) = h(X, Y).$$

□

Returning to the metric g, we can extend $\mathbb{C}$-linearly this metric on $TM^{\mathbb{C}}$. Then, it satisfies the following properties

(i) $g(\bar{Z}, \bar{W}) = \overline{g(Z, W)}$ for all $Z, W \in TM^{\mathbb{C}}$,

(ii) $g(Z, \bar{Z}) > 0$ for all $Z \neq 0$,

(iii) $g(Z, W) = 0$ for all $Z, W \in T^{1,0}M$, and all $Z, W \in T^{0,1}M$.

We can easily establish (i) and (ii). For (iii), when $Z, W \in T^{1,0}M$ write $Z = X - iJX$ and $W = Y - iJY$ in order to get

$$g(Z, W) = g(X - iJX, Y - iJY) = g(X, Y) - ig(JX, Y) - ig(X, JY) - g(JX, JY).$$

Since g is compatible with J, we have $g(JX, JY) = g(X, Y)$ and $g(JX, Y) = -g(X, JY)$. Therefore we get $g(Z, W) = 0$. In a similar fashion we obtain $g(Z, W) = 0$ for all $Z, W \in T^{0,1}M$.

Consider $\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \cdots, \frac{\partial}{\partial \bar{z}_n}\}$ as a basis for $TM^{\mathbb{C}}$. Let $g_{k\bar{l}}$ be the coefficients of the metric g with respect to the basis given above; here each $g_{k\bar{l}}$ is obtained by

$$g_{k\bar{l}} = g\left(\frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l}\right).$$

Proposition 1.5.2. *Let h be the positive hermitian form defined above. If we write $h_{kl} = h(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l})$, then the equality $h_{kl} = g_{k\bar{l}}$ holds.*

Proof. Since $\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right)$ and $\frac{\partial}{\partial \bar{z}_l} = \frac{1}{2}\left(\frac{\partial}{\partial x_l} + i\frac{\partial}{\partial y_l}\right)$, we have

$$
\begin{aligned}
g_{k\bar{l}} &= g\left(\frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right), \frac{1}{2}\left(\frac{\partial}{\partial x_l} + i\frac{\partial}{\partial y_l}\right)\right) \\
&= \frac{1}{4}\left(g\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right) - ig\left(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial x_l}\right) + ig\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_l}\right) + g\left(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l}\right)\right)
\end{aligned}
$$

But we also have $g(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}) = g(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_l})$ and $g(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_l}) = -g(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial x_l})$. This simplifies the above equality in

$$
\begin{aligned}
g_{k\bar{l}} &= \frac{1}{2}\left(g\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right) - ig\left(\frac{\partial}{\partial y_k}, \frac{\partial}{\partial x_l}\right)\right) \\
&= \frac{1}{2}\left(g\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right) - i\omega_g\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right)\right) \\
&= h_{kl}.
\end{aligned}
$$

□

On the other hand, due to that $\omega_g(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}) = -Im(h_{kl})$ and $\omega_g(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_l}) = g(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}) = Re(h_{kl})$, we can express ω_g as

$$
\omega_g = 2\sum_{k,l} Re(h_{kl})(dx_k \wedge dy_l) - 2\sum_{k<l} Im(h_{kl})(dx_k \wedge dx_l + dy_k \wedge dy_l). \qquad (1.2)
$$

Using the above formula and Proposition 1.5.2, we arrive to following result

Proposition 1.5.3. *The Kähler form ω_g can be expressed as*

$$
\omega_g = i\sum_{k,l} g_{k\bar{l}}\, dz_k \wedge d\bar{z}_l.
$$

Proof. Since $dz_k = x_k + iy_k$ and $d\bar{z}_l = dx_l - idy_l$, we obtain

$$
dz_k \wedge d\bar{z}_l = dx_k \wedge dx_l + dy_k \wedge dy_l + idy_k \wedge dx_l - idx_k \wedge dy_l.
$$

By a direct calculation we have

$$
\begin{aligned}
h_{kl} dz_k \wedge d\bar{z}_l = {} & Re(h_{kl})(dx_k \wedge dx_l + dy_k \wedge dy_l + idy_k \wedge dx_l - idx_k \wedge dy_l) \\
& + Im(h_{kl})(dx_k \wedge dx_l + dy_k \wedge dy_l + idy_k \wedge dx_l - idx_k \wedge dy_l) \\
= {} & Re(h_{kl})(dx_k \wedge dx_l + dy_k \wedge dy_l) + iRe(h_{kl})(dy_k \wedge dx_l - dx_k \wedge dy_l) \\
& + iIm(h_{kl})(dx_k \wedge dx_l + dy_k \wedge dy_l) - Im(h_{kl})(dy_k \wedge dx_l - dx_k \wedge dy_l).
\end{aligned}
$$

It is immediately evident that $\sum_{kl} Re(h_{kl})(dx_k \wedge dx_l + dy_k \wedge dy_l) = 0$. Furthermore, since $Im(h_{kl}) = -Im(h_{lk})$, we obtain

$$\sum_{k,l} Im(h_{kl})(dy_k \wedge dx_l - dx_k \wedge dy_l) = 0.$$

As a consequence, considering the last two results, we obtain the equation

$$\begin{aligned}\sum_{k,l} h_{kl} dz_k \wedge d\bar{z}_l = \sum_{k,l} & i(Re(h_{kl})(dy_k \wedge dx_l - dx_k \wedge dy_l) \\ & + Im(h_{kl})(dx_k \wedge dx_l + dy_k \wedge dy_l)).\end{aligned} \tag{1.3}$$

On the other hand, we have the equality

$$\sum_{k,l} Im(h_{kl})(dx_k \wedge dx_l + dy_k \wedge dy_l) = 2 \sum_{k<l} Im(h_{kl})(dx_k \wedge dx_l + dy_k \wedge dy_l).$$

Furthermore, by a calculation we obtain

$$\sum_{k,l} Re(h_{kl})(dy_k \wedge dx_l - dx_k \wedge dy_l) = -2 \sum_{k,l} Re(h_{kl})(dx_k \wedge dx_l).$$

Replacing these two last formulas in (1.3), we arrive to

$$\sum_{k,l} h_{kl} dz_k \wedge d\bar{z}_l = 2i \left(\sum_{k<l} Im(h_{kl})(dx_k \wedge dx_l + dy_k \wedge dy_l) - \sum_{k,l} Re(h_{kl})(dx_k \wedge dx_l) \right).$$

Multiplying both sides by i in the above equation, we obtain

$$i \sum_{k,l} h_{kl} dz_k \wedge d\bar{z}_l = 2 \sum_{k,l} Re(h_{kl})(dx_k \wedge dy_l) - 2 \sum_{k<l} Im(h_{kl})(dx_k \wedge dx_l + dy_k \wedge dy_l).$$

By (1.2), this means that $\omega_g = i \sum_{k,l} h_{kl} dz_k \wedge d\bar{z}_l$. Finally, using Proposition 1.5.2, we conclude that

$$\omega_g = i \sum_{k,l} g_{k\bar{l}} dz_k \wedge d\bar{z}_l.$$

□

Remark 1.5.4. On a Kähler manifold (M, g, J) we have that J is parallel with respect to the metric g; this means the identity $\nabla J = 0$, where ∇ represents the natural extension of the Levi-Civita connection on M to its correspondent vector bundle TM.

We notice easily that the Kähler form is a symplectic form. Indeed, since ω is closed, we only need to verify that ω_p is skew-symmetry and non-degenerate on T_pM . But this is easy: for $X, Y \in T_pM$ we have

$$\omega_g(X, Y) = g(JX, Y) = g(-X, JY) = -\omega_g(Y, X),$$

so ω_g is skew-symmetry. On the other hand, we have $\omega_g(X, Y) = 0$ for all $Y \in T_pM$ if and only if $g(JX, Y) = 0$ for all $Y \in T_pM$. Since g is an inner product, we have $JX = 0$, and so $X = 0$.

Remark 1.5.5. From the above, we have that all Kähler manifolds admit a symplectic form. However, the converse is not true. Thurston and Yamato gave examples of symplectic manifolds that do not admit a Kähler structure. See Cannas da Silva [9].

Example 1.5.6. [**The Fubini-Study metric**]. Next we define a Kähler form on $\mathbb{CP}^n$. This gives us a Kähler metric on $\mathbb{CP}^n$ which is called the *Fubini-Study metric*. Consider the open covering $\{U_k\}_{k=0}^n$ of $\mathbb{CP}^n$ and the maps $\varphi_k : U_k \to \mathbb{C}^n$ given in Example 1.4.2. On U_k we define the (1,1)-form

$$\omega_k = \frac{i}{2\pi}\partial\bar{\partial}\log\left(\sum_{l=0}^{n}\left|\frac{z_l}{z_k}\right|^2\right).$$

These ω_k define a global (1,1)-form on $\mathbb{CP}^n$. For this we must verify the condition $\omega_k|_{U_k\cap U_j} = \omega_j|_{U_k\cap U_j}$. In fact, we have

$$\begin{aligned}
\omega_k = \frac{i}{2\pi}\partial\bar{\partial}\log\left(\sum_{l=0}^{n}\left|\frac{z_l}{z_k}\right|^2\right) &= \frac{i}{2\pi}\partial\bar{\partial}\log\left(\sum_{l=0}^{n}\frac{|z_l|^2}{|z_j|^2}\frac{|z_j|^2}{|z_k|^2}\right)\\
&= \frac{i}{2\pi}\partial\bar{\partial}\log\left(\frac{|z_j|^2}{|z_k|^2}\sum_{l=0}^{n}\frac{|z_l|^2}{|z_j|^2}\right)\\
&= \frac{i}{2\pi}\partial\bar{\partial}\log\left(\left|\frac{z_j}{z_k}\right|^2\right) + \frac{1}{2\pi}\partial\bar{\partial}\log\left(\sum_{l=0}^{n}\left|\frac{z_l}{z_j}\right|^2\right)\\
&= \frac{i}{2\pi}\partial\bar{\partial}\log\left(\left|\frac{z_j}{z_k}\right|^2\right) + \omega_j.
\end{aligned}$$

Supposing $j > k$ and taking $w_j = \frac{z_j}{z_k}$, we get

$$\partial\bar{\partial}\log|w_j|^2 = \partial\bar{\partial}log(w_j\bar{w}_j) = \partial\left(\frac{w_j\bar{\partial}\bar{w}_j}{w_j\bar{w}_j}\right) = \frac{1}{\bar{w}_j}(-\bar{\partial}\partial\bar{w}_j) = 0.$$

Therefore $\omega_k = \omega_j$ on $U_k \cap U_j$. Next we show that ω_j is a real 2-form. Without loss of generality we can take $j = 0$. Thus, on U_0 we have $\varphi_0([z_0 : z_1 : \dots : z_n]) = \left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right)$. Taking coordinates $w_j = \frac{z_j}{z_k}$, we obtain

$$\omega_k = \frac{i}{2\pi}\partial\bar{\partial}\log\left(1 + \sum_{l=1}^{n} |w_l|^2\right),$$

and so

$$\begin{aligned}
\omega_k &= \frac{i}{2\pi}\partial\left(\frac{\sum_{l=1}^{n}\bar{\partial}(w_l\bar{w}_l)}{1 + \sum_{l=1}^{n}|w_l|^2}\right)\\
&= \frac{i}{2\pi}\partial\left(\frac{\sum_{l=1}^{n} w_l d\bar{w}_l}{1 + \sum_{l=1}^{n}|w_l|^2}\right)\\
&= \frac{i}{2\pi}\sum_{l=1}^{n}\partial\left(\frac{w_l d\bar{w}_l}{1 + \sum_{j=1}^{n}|w_j|^2}\right)\\
&= \frac{i}{2\pi}\sum_{l=1}^{n}\left(\frac{\left(1 + \sum_{j=1}^{n}|w_j|^2\right)dw_l - w_l\sum_{j=1}^{n}\bar{w}_j dw_j}{\left(1 + \sum_{j=1}^{n}|w_j|^2\right)^2}\right)\wedge d\bar{w}_l\\
&= \frac{i}{2\pi}\left(\frac{\sum_{l=1}^{n} dw_l d\bar{w}_l}{1 + \sum_{j=1}^{n}|w_j|^2} - \frac{\left(\sum_{j=1}^{n}\bar{w}_j dw_j\right)\wedge\left(\sum_{l=1}^{n} w_l d\bar{w}_l\right)}{\left(1 + \sum_{j=1}^{n}|w_j|^2\right)^2}\right).
\end{aligned}$$

Expressing w_l as $w_l = x_l + iy_l$ and $\bar{w}_l$ as $\bar{w}_l = x_l - iy_l$, we achieve

$$\begin{aligned}
\frac{\sum_{l=1}^{n} dw_l d\bar{w}_l}{1 + \sum_{j=1}^{n}|w_j|^2} &= \frac{\sum_{l=1}^{n}[(dx_l + idy_l)\wedge(dx_l - idy_l)]}{1 + \sum_{j=1}^{n}(x_j)^2 + (y_j)^2}\\
&= \frac{\sum_{l=1}^{n}[dx_l\wedge dx_l - 2idx_l\wedge dy_l + dy_l\wedge dy_l]}{1 + \sum_{j=1}^{n}(x_j)^2 + (y_j)^2}\\
&= -2i\frac{\sum_{l=1}^{n} dx_l\wedge dy_l}{1 + \sum_{j=1}^{n}(x_j)^2 + (y_j)^2}
\end{aligned}$$

and

$$\frac{\left(\sum_{k=1}^n \bar{w}_k dw_k\right) \wedge \left(\sum_{l=1}^n w_l d\bar{w}_l\right)}{\left(1 + \sum_{j=1}^n |w_j|^2\right)^2} =$$

$$\frac{\left(\sum_{k=1}^n (x_k - iy_k)(dx_k + idy_k)\right) \wedge \left(\sum_{l=1}^n (x_l + iy_l)(dx_l - idy_l)\right)}{\left(1 + \sum_{j=1}^n (x_j)^2 + (y_j)^2\right)^2}$$

$$= \frac{\left[\sum_{k=1}^n (x_k - iy_k)dx_k + (y_k + ix_k)dy_k\right] \wedge \left[\sum_{l=1}^n (x_l + iy_l)dx_l + (y_l - ix_l)dy_l\right]}{\left(1 + \sum_{j=1}^n (x_j)^2 + (y_j)^2\right)^2}$$

$$= \frac{\sum_{k,l=1}^n (x_k - iy_k)(x_l + iy_l)dx_k \wedge dx_l + \sum_{k,l=1}^n (y_k + ix_k)(y_l - ix_l)dy_k \wedge dy_l}{\left(1 + \sum_{j=1}^n (x_j)^2 + (y_j)^2\right)^2} +$$
$$\frac{\sum_{k,l=1}^n (x_k - iy_k)(y_l - ix_l)dx_k \wedge dy_l + \sum_{k,l=1}^n (y_k + ix_k)(x_l + iy_l)dy_k \wedge dx_l}{\left(1 + \sum_{j=1}^n (x_j)^2 + (y_j)^2\right)^2}$$

$$= \frac{\sum_{k,l=1}^n (x_kx_l + y_ky_l + i(x_ky_l - x_ly_k))dx_k \wedge dx_l}{\left(1 + \sum_{j=1}^n (x_j)^2 + (y_j)^2\right)^2} + \frac{\sum_{k,l=1}^n (y_ky_l + x_kx_l + i(x_ky_l - x_ly_k))dy_k \wedge dy_l}{\left(1 + \sum_{j=1}^n (x_j)^2 + (y_j)^2\right)^2} +$$
$$\frac{\sum_{k,l=1}^n (x_k - iy_k)(y_l - ix_l)dx_k \wedge dy_l - \sum_{k,l=1}^n (y_l + ix_l)(x_k + iy_k)dx_k \wedge dy_l}{\left(1 + \sum_{j=1}^n (x_j)^2 + (y_j)^2\right)^2}$$

$$= \frac{\sum_{k,l=1}^n i(x_ky_l - x_ly_k)dx_k \wedge dx_l + \sum_{k,l=1}^n i(x_ky_l - x_ly_k)dy_k \wedge dy_l}{\left(1 + \sum_{j=1}^n (x_j)^2 + (y_j)^2\right)^2} - 2i\frac{\sum_{k,l=1}^n (x_kx_l + y_ky_l)dx_k \wedge dy_l}{\left(1 + \sum_{j=1}^n (x_j)^2 + (y_j)^2\right)^2}$$

$$= 2i\,\frac{\sum_{k,l=1}^n x_ky_l dx_k \wedge dx_l + \sum_{k,l=1}^n x_ky_l dy_k \wedge dy_l - \sum_{k,l=1}^n (x_kx_l + y_ky_l)dx_k \wedge dy_l}{\left(1 + \sum_{j=1}^n (x_j)^2 + (y_j)^2\right)^2}.$$

Putting together both expressions we get

$$\omega_k = \frac{1}{\pi}\frac{\sum_{l=1}^n dx_l \wedge dy_l}{1 + \sum_{j=1}^n (x_j)^2 + (y_j)^2} + \frac{1}{\pi}\frac{\sum_{k,l=1}^n x_ky_l dx_k \wedge dx_l + \sum_{k,l=1}^n x_ky_l dy_k \wedge dy_l - \sum_{k,l=1}^n (x_kx_l + y_ky_l)dx_k \wedge dy_l}{\left(1 + \sum_{j=1}^n (x_j)^2 + (y_j)^2\right)^2}.$$

Thus, ω is a real 2-form and (1, 1)-form. Using Proposition 1.4.12 we conclude that ω is also a closed form.

On the other hand, to show that ω is Kähler, we first express ω as in

Proposition 1.5.3. In fact, we have

$$\omega = \frac{i}{2\pi}\left(\frac{\sum_{l=1}^{n} dw_l \wedge d\bar{w}_l}{1+\sum_{j=1}^{n}|w_j|^2} - \frac{(\sum_{j=1}^{n} \bar{w}_j dw_j) \wedge (\sum_{l=1}^{n} w_l d\bar{w}_l)}{\left(1+\sum_{j=1}^{n}|w_j|^2\right)^2} \right)$$

$$= \frac{i}{2\pi}\left(\frac{\sum_{l=1}^{n} dw_l \wedge d\bar{w}_l}{1+\sum_{j=1}^{n}|w_j|^2} - \frac{\sum_{k,l=1}^{n} \bar{w}_k w_l dw_k \wedge d\bar{w}_l}{\left(1+\sum_{j=1}^{n}|w_j|^2\right)^2} \right)$$

$$= \frac{i}{2\pi}\left(\sum_{k,l=1}^{n} \frac{\delta_{kl}\left(1+\sum_{j=1}^{n}|w_j|^2\right)}{\left(1+\sum_{j=1}^{n}|w_j|^2\right)^2} dw_k \wedge d\bar{w}_l - \frac{\sum_{k,l=1}^{n} \bar{w}_k w_l dw_k \wedge d\bar{w}_l}{\left(1+\sum_{j=1}^{n}|w_j|^2\right)^2} \right)$$

$$= \frac{i}{2\pi}\left(\sum_{k,l=1}^{n} \frac{\delta_{kl}\left(1+\sum_{j=1}^{n}|w_j|^2\right) - \bar{w}_k w_l}{\left(1+\sum_{j=1}^{n}|w_j|^2\right)^2} \right)$$

Taking $g_{k\bar{l}} = \delta_{kl}(1+\sum_{j=1}^{n}|w_j|^2) - \bar{w}_k w_l$, we see that $[g_{k\bar{l}}]$ is positive definite and Hermitian. Since we have

$$[g_{k\bar{l}}] = (1+\sum_{j=1}^{n}|w_j|^2)I_n - \bar{w}.w^t,$$

for $u \neq 0$, we get

$$u^t[g_{k\bar{l}}]\bar{u} = u^t\left((1+\sum_{j=1}^{n}|w_j|^2)I_n - \bar{w}.w^t\right)\bar{u}$$

$$= (1+\sum_{j=1}^{n}|w_j|^2)\sum_{l=1}^{n}|u_l|^2 - u^t(\bar{w}w^t)\bar{u}$$

$$= (u,u) + (u,u)(w,w) - (u,w)(w,u)$$

$$= (u,u) + (u,u)(w.w) - |(w,u)|^2 .$$

Then, from $(u,u) > 0$ and $(u,u)(w,w) \geq |(w,u)|^2$, we obtain $u^t[g_{k\bar{l}}]\bar{u} > 0$. Thus ω is the Kähler form of a Kähler metric g. This metric g is known as the *Fubini-Study metric.*

Example 1.5.7. In the example above, for $\mathbb{CP}^1$ the Kähler form $\omega_{FS} = \omega$ in real coordinates is given by

$$\omega_{FS} = \frac{1}{\pi}\frac{dx \wedge dy}{(1+x^2+y^2)^2}.$$

Since $S^2 \cong \mathbb{CP}^1$, we have a Kähler form for S^2, which is represented by ω_{FS} as well. Let us see how this Kähler form is compatible with the complex structure defined on S^2 in Example 1.4.4. Easily we verify for the local basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ the equality $\frac{\partial}{\partial x} = J\frac{\partial}{\partial y}$. Thus, we have

$$\omega_{FS}(J\frac{\partial}{\partial x}, J\frac{\partial}{\partial y}) = \omega_{FS}(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}),$$

and, (S^2, J, ω_{FS}) is a Kähler manifold.

Let (M, g, J) be a Kähler manifold and ∇ be its Levi-Civita connection. Given local coordinates $(z_1, \ldots, z_n)$, we have a local basis $\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}\}$ of $TM^{\mathbb{C}}$, we can extend $\mathbb{C}$-linearly ∇ to $TM^{\mathbb{C}}$. Then we define the Christoffel symbols Γ^m_{kl} by

$$\nabla_{\frac{\partial}{\partial z_k}} \frac{\partial}{\partial z_l} = \Gamma^m_{kl}\frac{\partial}{\partial z^m} + \Gamma^{\bar{m}}_{kl}\frac{\partial}{\partial \bar{z}^m},$$

$$\nabla_{\frac{\partial}{\partial z_k}} \frac{\partial}{\partial \bar{z}_l} = \Gamma^m_{k\bar{l}}\frac{\partial}{\partial z^m} + \Gamma^{\bar{m}}_{k\bar{l}}\frac{\partial}{\partial \bar{z}^m}.$$

Using the equality $\nabla J = 0$, we have

$$\nabla_{\frac{\partial}{\partial z_k}} (J\frac{\partial}{\partial z_l}) = J(\nabla_{\frac{\partial}{\partial z_k}} \frac{\partial}{\partial z_l}).$$

Next we get

$$\nabla_{\frac{\partial}{\partial z_k}}(J\frac{\partial}{\partial z_l}) = \nabla_{\frac{\partial}{\partial z_k}}(i\frac{\partial}{\partial z^l}) = i(\Gamma^m_{kl}\frac{\partial}{\partial z^m} + \Gamma^{\bar{m}}_{kl}\frac{\partial}{\partial \bar{z}^m})$$

and

$$J(\nabla_{\frac{\partial}{\partial z_k}} \frac{\partial}{\partial z_l}) = J(\Gamma^m_{kl}\frac{\partial}{\partial z_m} + \Gamma^{\bar{m}}_{kl}\frac{\partial}{\partial \bar{z}_m}) = i(\Gamma^m_{kl}\frac{\partial}{\partial z_m} - \Gamma^{\bar{m}}_{kl}\frac{\partial}{\partial \bar{z}_m});$$

so $\Gamma^{\bar{m}}_{kl} = 0$ is satisfied.

On the other hand, because of

$$\nabla_{\frac{\partial}{\partial z_k}} (J\frac{\partial}{\partial \bar{z}_l}) = J(\nabla_{\frac{\partial}{\partial z_k}} \frac{\partial}{\partial \bar{z}_l}),$$

we obtain

$$\nabla_{\frac{\partial}{\partial z_k}} (J\frac{\partial}{\partial \bar{z}^l}) = -i(\Gamma^m_{k\bar{l}}\frac{\partial}{\partial z^m} + \Gamma^{\bar{m}}_{k\bar{l}}\frac{\partial}{\partial \bar{z}^m})$$

and

$$J(\nabla_{\frac{\partial}{\partial z_k}} \frac{\partial}{\partial \bar{z}_l}) = i(\Gamma^m_{k\bar{l}}\frac{\partial}{\partial z^m} - \Gamma^{\bar{m}}_{k\bar{l}}\frac{\partial}{\partial \bar{z}^m});$$

and we get $\Gamma^m_{k\bar{l}} = 0$.

If α, β, λ run over the indices $\{1, 2, \ldots, n, \bar{1}, \bar{2}, \ldots, \bar{n}\}$, we have $\overline{\Gamma^\lambda_{\alpha\beta}} = \Gamma^{\bar{\lambda}}_{\bar{\alpha}\bar{\beta}}$ and $\Gamma^\lambda_{\alpha\beta} = \Gamma^\lambda_{\beta\alpha}$. Therefore, we conclude that $\Gamma^m_{\bar{k}l}, \Gamma^m_{k\bar{l}}, \Gamma^{\bar{m}}_{kl}$ and $\Gamma^m_{\bar{k}\bar{l}}$ are all 0. Hence the only surviving terms are Γ^m_{kl} and $\Gamma^{\bar{m}}_{\bar{k}\bar{l}}$.

About the *Riemannian curvature* R on TM given by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

we can extend it again in a $\mathbb{C}$-linear way to $TM^{\mathbb{C}}$. If so, given the local complex coordinates $z_1, ..., z_n$, we define the terms $R_{i\bar{j}k\bar{l}}$ by

$$R_{i\bar{j}k\bar{l}} = R^{\nabla}\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial z_k}, \frac{\partial}{\partial \bar{z}_l}\right),$$

where R^{∇} is the Riemannian curvature tensor, which is defined as

$$R^{\nabla}(X,Y,Z,W) = g(R(X,Y)Z,W).$$

The *Ricci curvature* is defined as the trace of $R_{i\bar{j}k\bar{l}}$. Therefore, if we express the Kähler form ω_g as

$$\omega_g = \frac{i}{2}\sum_{k,l=1} g_{k\bar{l}}\, dz_k \wedge d\bar{z}_l,$$

we arrive to following expression for the Ricci curvature

$$Ric_{k\bar{l}} = g_{i\bar{j}} R_{i\bar{j}k\bar{l}} = -\frac{\partial^2}{\partial z_k \partial \bar{z}_l}(\log(det(g_{i\bar{j}})).$$

Finally, we say that a Kähler metric g is *Kähler-Einstein* if $Ric_{i\bar{j}} = \lambda g_{i\bar{j}}$ for some $\lambda \in \mathbb{R}$.

Example 1.5.8. On $\mathbb{CP}^1$, the metric ω_{FS} is Kähler-Einstein. Indeed, we have

$$Ric_{1\bar{1}} = -\frac{\partial^2}{\partial z \partial \bar{z}}(\log(\det(1+|z|^2)^{-2})) = 2g_{1\bar{1}}.$$

Here $\lambda = 2$ is the so-called *Einstein constant*. In general, on $\mathbb{CP}^n$ we have

$$Ric_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} log\left(\frac{1}{(1+|z|^2)^{n+1}}\right) = (n+1)g_{i\bar{j}}.$$

1.6 Contact structures

Let M be a smooth manifold of dimension $2n+1$. Let η a 1-form such that $\eta \wedge (d\eta)^n \neq 0$, we say that (M, η) is a *contact manifold*. The 1-form η is then called the *contact form*.

Example 1.6.1. On $\mathbb{R}^{2n+1}$ we define a 1-form

$$\eta = dz - \sum_{i=1}^{n} y_i dx_i,$$

where $(x_1, x_2, ..., x_n, y_1, ..., y_n, z)$ represent the cartesian coordinates on $\mathbb{R}^{2n+1}$. Clearly $(\mathbb{R}^{2n+1}, \eta)$ is a contact manifold.

Locally we can always find coordinates which allow us to define the contact form η in a similar way to the example given above.

Proposition 1.6.2 (Darboux)**.** *Let (M, η) be a contact manifold. Around each $p \in M$ there exit local coordinates $x_1, ..., x_n, y_1, ..., y_n, z$ in $U \subset M$ such that*

$$\eta_U = dz - \sum_{i=1}^{n} y_i dx_i.$$

Proof. See Blair [6]. □

Definition 1.6.3. Let (M, η) be a contact manifold. Given an element $\xi \in \mathfrak{X}(M)$ we say that ξ is a *Reeb vector field* if it satisfies

(1) $d\eta(\xi, \cdot) = 0$,

(2) $\eta(\xi) = 1$.

The following theorem guarantees the existence and uniqueness of the Reeb vector field.

Proposition 1.6.4. *On a contact manifold (M, η) there exists a unique vector field ξ that satisfies conditions (1), (2) above.*

Proof. See Blair [6]. □

From the formula for the Lie derivative $\mathcal{L}_\xi = d \circ (\xi \lrcorner) + (\xi \lrcorner \circ d)$, where $\lrcorner$ is the interior product on M, we have

$$\mathcal{L}_\xi \eta(X) = X(\eta(\xi)) + d\eta(\xi, X) = 0.$$

Thus we get $\mathcal{L}_\xi \eta = 0$. On the other hand, as $\mathcal{L}_\xi d\eta = d(\mathcal{L}_\xi \eta)$ holds, we obtain $\mathcal{L}_\xi d\eta = 0$.

Proposition 1.6.5. *Let $i : M \to \mathbb{R}^{2n+2}$ be a smooth hypersurface immersed in $\mathbb{R}^{2n+2}$ and suppose that no tangent space of M contains the origin of $\mathbb{R}^{2n+2}$. Then M admits a contact form.*

Proof. See Blair [6]. □

As consequence of above proposition, we notice that S^{2n+1} carries a contact form. Afterwards,for our purposes we use another contact structure on S^{2n+1}, which is obtained via the Boothby-Wang fibration.

Next, we study briefly the associated metric structure on a contact manifold (M, η).

Definition 1.6.6. Let M be a smooth manifold of dimension $2n + 1$. We say M has an *almost contact structure* if there exists a (1,1)-tensor field φ, a vector field ξ, and a 1-form η related by

(1) $\varphi^2 = -id + \eta \otimes \xi$,

(2) $\eta(\xi) = 1$.

In this case we say $(M ; \varphi, \xi, \eta)$ is an *almost contact manifold.*

Proposition 1.6.7. *If* $(M ; \varphi, \xi, \eta)$ *is an almost contact manifold, then* $\varphi\xi = 0$ *and* $\eta \circ \varphi = 0$.

Proof. See Blair [6]. □

We say that a Riemannian metric g is *compatible* with an almost contact structure (φ, ξ, η) on M if it satisfies the condition

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y).$$

In this case (φ, η, ξ, g) is an *almost contact metric structure on* M.

Proposition 1.6.8. *Let* (M, η) *be a contact manifold and* ξ *be its Reeb vector field. Then* M *admits a* (1,1)*-tensor field* φ *and a Riemannian metric* g *such that* (φ, ξ, η, g) *defines an almost contact metric structure on* M *subject to* $g(X, \varphi Y) = d\eta(X, Y)$.

Proof. See Blair [6]. □

In the case when a contact manifold (M, η) admits an almost contact metric structure (φ, η, ξ, g) that satisfies the equation above, we say that $(M ; \varphi, \eta, \xi, g)$ is a *contact metric manifold* and g is an *associated metric* on M.

Remark 1.6.9. If (φ, η, ξ, g) is an almost contact metric structure on M, then we obtain

$$
\begin{aligned}
0 = g(\varphi X, \varphi\xi) &= g(X, \xi) - \eta(X)\eta(\xi) \\
&= g(X, \xi) - \eta(X);
\end{aligned}
$$

hence we get $\eta(X) = g(X, \xi)$.

Let M be a smooth manifold endowed with an almost contact metric structure (φ, η, ξ, g). We define a 2-form Ω on M by

$$\Omega(X, Y) = g(X, \varphi Y).$$

This 2-form Ω is called the *fundamental 2-form* of the almost contact metric structure (φ, ξ, η, g).

Remark 1.6.10. Notice that Ω is skew-symmetry. In fact, we obtain

$$
\begin{aligned}
\Omega(X, Y) = g(X, \varphi Y) &= g(\varphi X, \varphi^2 Y) + \eta(X)\eta(\varphi Y) \\
&= g(\varphi X, -Y + \eta(Y)\xi) \\
&= -g(\varphi X, Y) + \eta(Y)g(\varphi X, \xi).
\end{aligned}
$$

As we have $\eta(Y)g(\varphi X, \xi) = \eta(Y)[g(\varphi^2 X, \varphi X) + \eta(\varphi X)\eta(\xi)] = 0$, last formula is equivalent to $\Omega(X, Y) = -\Omega(Y, X)$.

Remark 1.6.11. Since $\Omega(X, Y) = g(X, \varphi Y)$, we have that (φ, η, ξ, g) is a contact metric structure on a contact manifold (M, η) whenever $\Omega = d\eta$.

Remark 1.6.12. On a manifold M with a contact metric structure (φ, ξ, η, g), the integral curves are geodesics. To show that we remember that the Lie derivative satisfies the Leibniz rule. It means

$$\mathsf{L}_\xi(\eta(x)) = \xi\eta(X) - \eta(\mathsf{L}_\xi X).$$

Then, since $\mathsf{L}_\xi\eta = 0$ and $\mathsf{L}_\xi X = [\xi, X]$, we have

$$
\begin{aligned}
0 = \mathsf{L}_\xi\eta(X) = \xi(\eta(X)) - \eta([\xi, X]) &= \xi(g(X, \xi)) - g([\xi, X], \xi) \\
&= \xi(g(X, \xi)) - g(\nabla_\xi X - \nabla_X\xi, \xi) \\
&= \xi(g(X, \xi)) - g(\nabla_\xi X, \xi) + g(\nabla_X\xi, \xi).
\end{aligned}
$$

As $g(\nabla_\xi X, \xi) = \xi(g(X, \xi)) - g(X, \nabla_\xi\xi)$ and $0 = X(g(\xi, \xi)) = 2g(\nabla_X\xi, \xi)$, we obtain $0 = \mathsf{L}_\xi\eta(X) = g(X, \nabla_\xi\xi)$; but this means $\nabla_\xi\xi = 0$.

Definition 1.6.13. A vector field ξ is a *Killing vector field* if ξ generates a 1-parameter group of isometries.

There is a helpful characterization of this concept (see Moroianu [23] for a proof):

Lemma 1.6.14. *A vector field ξ on a Riemannian manifold (M,g) is a Killing vector field if one of the following conditions holds.*

(1) The Lie derivative of g with respect to ξ vanishes, that is $\mathrm{L}_\xi g = 0$.

(2) The covariant derivative $\nabla\xi$ verifies the equality

$$g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 0.$$

Proof. See Moroianu [23]. □

Definition 1.6.15. Take a contact metric structure (φ, ξ, η, g) on M. If ξ is a Killing vector field, we say that (φ, ξ, η, g) is a *K-contact structure.* In this case, M is called *K-contact manifold.*

As we shall see later, the Hopf fibration has as vertical vector field the Reeb vector field associated to the contact structure on S^3 induced by the Fubini-Study form. Furthermore, this vertical vector field is Killing.

Chapter 2

Principal bundles

2.1 Connection and curvature on principal bundles

In this section we expose some results on principal bundles. Some propositions and theorems are taken from Kobayashi [17] and Figueroa [11].

Let M and P be smooth manifolds, $\pi : P \to M$ a smooth surjective map (a *projection*), and G a Lie group acting on P on the right. We say that (P, G, M) is a *principal G-bundle* if

i) G acts freely on P (this means that the stabilizer of each element of P contains only the identity of G);

ii) there exists an open cover $\{U_\alpha\}$ of M and a collection of local diffeomorphism

$$\begin{aligned} \psi_\alpha = (\pi, g_\alpha): \quad \pi^{-1}(U_\alpha) &\longrightarrow U_\alpha \times G \\ p &\longrightarrow \psi_\alpha(p) = (\pi(p), g_\alpha(p)), \end{aligned}$$

for which the following diagram

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "\psi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times G \arrow[dl, "pr_1"] \\ & U_\alpha & \end{tikzcd}$$

commutes; here each g_α is a G-equivariant map, i.e., satisfies $g_\alpha(ph) = g_\alpha(p)h$ for each $p \in \pi^{-1}(U_\alpha)$ and $h \in G$.

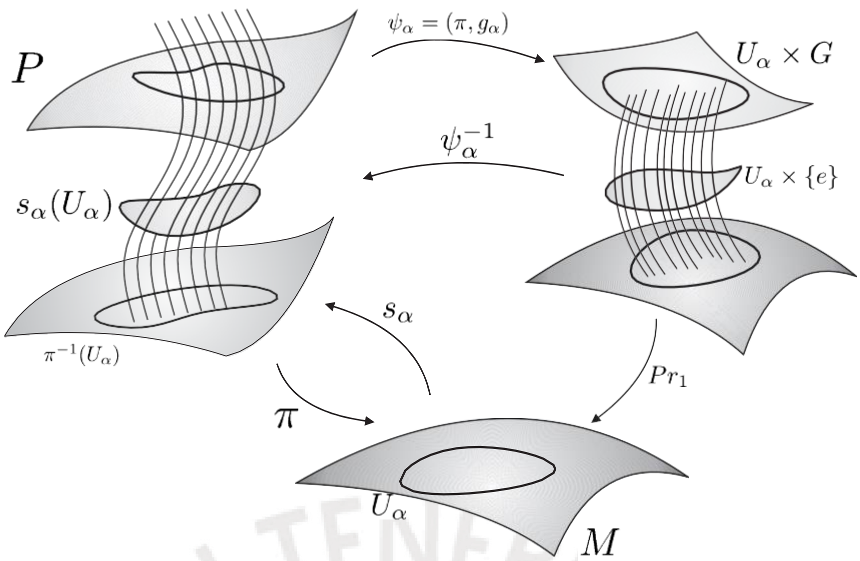

Figure 2.1: Here ψ_a maps $s_a(U_a)$ on $U_a \times \{e\}$. Furthermore, for each element $m \in U_a$ we have $\psi_a(s_a(m)) = (m, e)$.

The collection of pairs (U_a, ψ_a) is called *trivialization* of the principal bundle $\pi : P \to M$. Notice that for each element $m \in U_a$, the fiber $\pi^{-1}(m)$ has a group structure isomorphic to G.

A *local section* on the principal bundle $\pi : P \to M$ is a smooth map $s : U \subset M \to P$ such that $\pi \circ s = id_U$.

For each trivialization (U_a, ψ_a) of the principal S^1-bundle $\pi : P \to M$, we define the maps $s_a : U_a \to \pi^{-1}(U_a)$ as $s_a(m) = \psi_a^{-1}(m, e)$ for all $m \in U_a$. Clearly these maps s_a are local sections on P; and they are called *canonical sections*. Compare Figure 2.1. Using these canonical sections s_a, we write each element $p \in \pi^{-1}(m)$ as $p = s_a(m)g_a(p)$.

Remark 2.1.1. If we can obtain a global section on the principal G-bundle $\pi : P \to M$, or which is equivalent to define a smooth map s on M such that $\pi \circ s = id_M$, then the principal G-bundle is trivial. In fact, in such case we can define a global trivialization ψ as $\psi^{-1}(m, g) = s(m)g = p$. Then ψ is given by $\psi(p) = (m, g)$.

Let (U_a, ψ_a) be a trivialization of a principal bundle $\pi : P \to M$. If we take another trivialization (U_β, ψ_β) of P, then we detect a variation on the fibers with respect to the other trivialization. Taking $m \in U_{a\beta} = U_a \cap U_\beta$ and

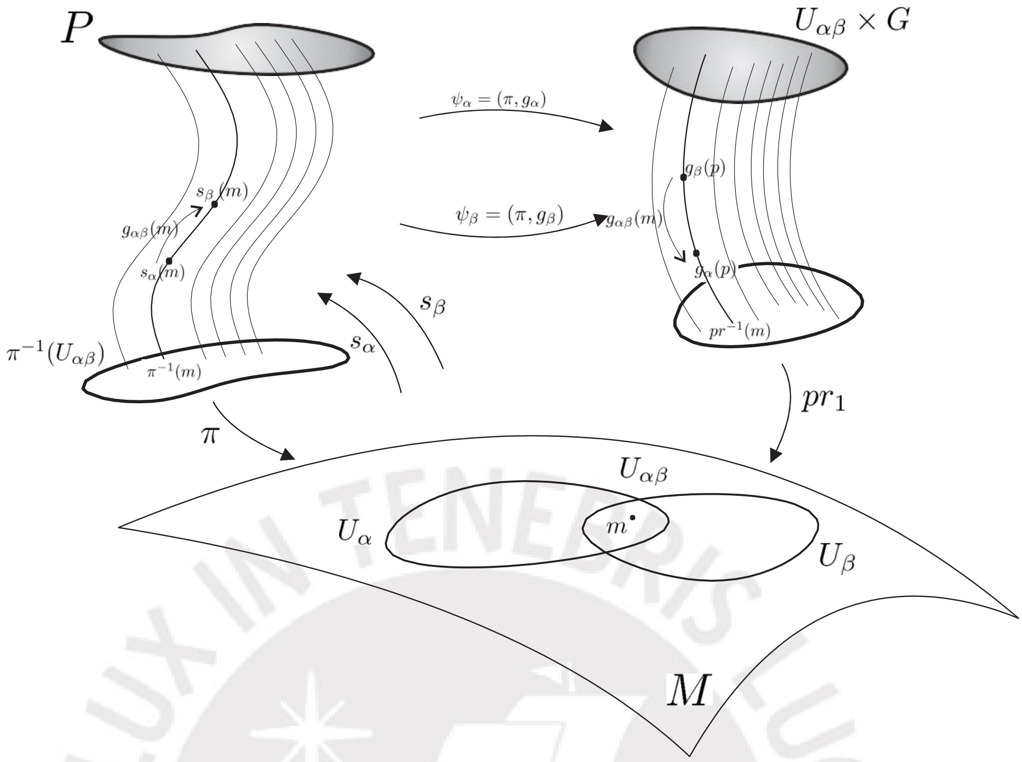

Figure 2.2: The variation between $s_\alpha(m)$ and $s_\beta(m)$ on the fiber is the inverse with respect to the variation between $g_\beta(p)$ and $g_\alpha(p)$ in G for any $p \in \pi^{-1}(m)$.

$p \in \pi^{-1}(m)$, we can write this variation as $g_{\alpha\beta}(p)$. We notice that $g_{\alpha\beta}$ can be expressed as $g_{\alpha\beta} = g_\alpha g_\beta^{-1}(p)$(see Figure 2.2.). In fact, as we try to explain next, this variation $g_{\alpha\beta}(p)$ is constant throughout the fiber $\pi^{-1}(m)$.
Let $p \in \pi^{-1}(m)$ and $h \in G$. Then we have

$$g_{\alpha\beta}(ph) = g_\alpha(ph)(g_\beta(ph))^{-1} = g_\alpha(p)hh^{-1}(g_\beta(p))^{-1} = g_{\alpha\beta}(p).$$

Due to above circumstance, we can say that the variation $g_{\alpha\beta}$ is independent of the element of the fiber. It allows us to define the following map with certain abuse of notation

$$\begin{aligned} g_{\alpha\beta}: \quad U_{\alpha\beta} &\longrightarrow G \\ m &\longrightarrow g_{\alpha\beta}(m) = g_{\alpha\beta}(p), \end{aligned}$$

where $p \in \pi^{-1}(m)$. These maps $g_{\alpha\beta}$ are called *transition maps*. Moreover these verify

$$g_{\alpha\beta}(m)g_{\beta\alpha}(m) = e,$$
$$g_{\alpha\beta}(m)g_{\beta\gamma}(m)g_{\gamma\alpha}(m) = e;$$

called *cocycle conditions*. Conversely, cocycle conditions are enough to induce the structure of a principal G-bundle on P .

Proposition 2.1.2. *For each $m \in U_{\alpha\beta}$ we have*

$$s_\beta(m) = s_\alpha(m)g_{\alpha\beta}(m).$$

Proof. With $p \in \pi^{-1}(m)$, we have $p = s_\beta(m)g_\beta(p) = s_\alpha(m)g_\alpha(p)$. So we get

$$s_\beta(m) = p(g_\beta(p))^{-1} = s_\alpha(m)g_\alpha(p)(g_\beta(p))^{-1} = s_\alpha(m)g_{\alpha\beta}(m).$$

□

Example 2.1.3. [**Hopf fibration.**] Sonsider $S^{2n+1} \subset \mathbb{C}^{n+1}$ and take an element $(z_0, ..., z_n) \in S^{2n+1}$. Then we have $|z_0|^2 + ... + |z_n|^2 = 1$. We define the projection map $\pi : S^{2n+1} \to \mathbb{CP}^n$ by $\pi(z_0, ..., z_n) = [z_0 : ... : z_n]$. It is easy to see that π is smooth and onto. Let us show that $\pi : S^{2n+1} \to \mathbb{CP}^n$ is a principal S^1-bundle. First, we need to define the right action of S^1 on S^{2n+1} by $(z_0, ..., z_n)e^{i\theta} = (z_0e^{i\theta}, ..., z_ne^{i\theta})$. This action is well-defined: indeed, if $(z_0, ..., z_n) \in S^{2n+1}$, we get

$$|e^{i\theta}z_0|^2 + ... + |e^{i\theta}z_n|^2 = |z_0|^2 + ... + |z_n|^2 = 1.$$

On the other hand, S^1 acts freely on S^{2n+1}. In fact, taking $(z_0, ..., z_n)$ in S^{2+1} and $e^{i\theta}$ in S^1 such that $(z_0, ..., z_n)e^{i\theta} = (z_0, ..., z_n)$, we get $e^{i\theta} = 1$. Furthermore, we notice that $\pi^{-1}([z_0 : ... : z_n])$ is isomorphic to S^1. Considering $[z_0, ..., z_n]$ in $\mathbb{CP}^n$ so that $|z_0|^2 + ... + |z_n|^2 = 1$, then for any $(w_0, ..., w_n) \in \pi^{-1}([z_0, ..., z_n])$, we have $(w_0, ..., w_n) = \lambda(z_0, ..., z_n)$ for some $\lambda \in \mathbb{C}^*$. Since $(w_0, ..., w_n)$ belongs to S^{2n+1}, we get

$$1 = |w_0|^2 + ... + |w_n|^2 = |\lambda|^2(|z_0|^2 + ... + |z_n|^2) = |\lambda|^2.$$

Thus, λ is an element of S^1. The isomorphism between S^1 and $\pi^{-1}([z_0 : ... : z_n])$ is given by $\lambda \mapsto \lambda(z_0, ..., z_n)$. It gives us the idea that S^{2n+1} is foliated by great circles.

For the existence of trivializations, we take the open covering $\{U_k\}_{k=0}^n$ of $\mathbb{CP}^n$ given by Example 1.4.2. Then we define ψ_k as

$$\begin{aligned} \psi_k : \quad \pi^{-1}(U_k) &\longrightarrow U_k \times S^1 \\ (z_0, ..., z_n) &\longmapsto \psi_k(z_0, ..., z_n) = ([z_0 : ... : z_n], e^{i\theta}), \end{aligned}$$

where $z_k = |z_k|e^{i\theta}$.

Example 2.1.4. Taking $n = 1$ in the example above, we obtain the principal S^1-bundle $\pi : S^3 \to \mathbb{C}P^1$. Using the identification $\mathbb{C}P^1 \cong S^2$, we get a principal S^1-bundle over S^2. (Since we can identify $U_0 = \mathbb{C}P^1 - \{[1:0]\}$ with $\mathbb{R}^2$ by $[z_0 : z_1] \mapsto \frac{z_0}{z_1} = x + iy = (x, y)$, and $\mathbb{R}^2$ with $S^2 - \{(0, 0, 1)\}$ by means of the stereographic projection

$$(x, y) \mapsto \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{x^2 + y^2 - 1}{1 + x^2 + y^2} \right),$$

we arrive to $\mathbb{C}P^1 - \{[1 : 0]\} \cong S^2 - \{(0, 0, 1)\}$. In the end, identifying $[1 : 0]$ with $(0, 0, 1)$ we get $\mathbb{C}P^1 \cong S^2$.) Next, if we take $z_0 = m + in$ and $z_1 = p + iq$, we can express an element (z_0, z_1) in S^3 as (m, n, p, q). Then the projection $\pi : S^3 \to S^2$ is given by

$$\pi(m, n, p, q) = \pi(z_0, z_1) = [z_0 : z_1] = \frac{z_1}{z_0} = Re\left(\frac{z_1}{z_0}\right) + iIm\left(\frac{z_1}{z_0}\right).$$

If we calculate the real and imaginary part of $\frac{z_1}{z_0}$, we obtain $Re\left(\frac{z_1}{z_0}\right) = Re\left(\frac{z_1 \bar{z}_0}{|z_0|^2}\right) = \frac{pm + nq}{p^2 + q^2}$ and $Im\left(\frac{z_1}{z_0}\right) = Im\left(\frac{z_1 \bar{z}_0}{|z_0|^2}\right) = \frac{mq - np}{p^2 + q^2}$. Finally, using the inverse map of the stereographic projection, we reach

$$\pi(m, n, p, q) = (2pm + 2qn, 2mq - np, p^2 + q^2 - m^2 - n^2).$$

Let $\pi : P \to M$ be a principal G-bundle. For each $p \in \pi^{-1}(m) \subset P$, where $m \in M$, we consider the map

$$\begin{aligned} \sigma_p : \quad G &\longrightarrow \pi^{-1}(m) \subset P \\ g &\longrightarrow \sigma_p(g) = pg. \end{aligned}$$

Since the action of G over P is free, the derivative of σ_p in e, i.e.,

$$(\sigma_p)_{*,e} : \mathfrak{g} \to T_p(\pi^{-1}(m)) \subset T_pP,$$

is an isomorphism. Then each $X \in \mathfrak{g}$ defines a tangent vector to $\pi^{-1}(m)$ in p. Thus, each $X \in \mathfrak{g}$ induces a vector field X^* on P, called *fundamental vector field*, naturally defined by

$$X^*(p) = (\sigma_p)_{*,e}(X).$$

When fixing $f \in C^\infty(P)$, we obtain

$$\begin{aligned} X^*(p)(f) = (\sigma_p)_{*,e}(X)(f) = X(f \circ \sigma_p) &= \frac{d}{dt}(\exp(tX)) \mid_{t=0} (f \circ \sigma_p) \\ &= \frac{d}{dt} f(p \exp(tX)) \mid_{t=0} \\ &= \frac{d}{dt}(p \exp(tX)) \mid_{t=0} (f), \end{aligned}$$

so we have

$$X^*(p) = \frac{d}{dt}(p \exp(tX)) \mid_{t=0}.$$

Furthermore, the assignement $X \mapsto X^*$ is an isomorphism of Lie algebras over its image. As consequence, we have the following proposition.

Proposition 2.1.5. *Let X^*, Y^* be the fundamental vector fields induced by $X, Y \in \mathfrak{g}$, respectively. Then $[X^*, Y^*] = [X, Y]^*$.*

Proof. Since $\sigma_{*,e}$ is an isomorphism of Lie algebras, we have

$$[X^*, Y^*] = [\sigma_{*,e}(X), \sigma_{*,e}(Y)] = \sigma_{*,e}([X, Y]) = [X, Y]^*.$$

□

Proposition 2.1.6. *Let X^* be the fundamental vector field induced by $X \in \mathfrak{g}$. Then we have*

$$(R_g)_*(X^*) = (Ad_{g^{-1}}X)^*.$$

Proof. For $p \in P$ and $f \in C^\infty(P)$, we have

$$\begin{aligned} (R_g)_*(X^*)_p(f) = X^*_p(f \circ R_g) &= \frac{d}{dt}(f \circ R_g(p \exp(tX))) \mid_{t=0} \\ &= \frac{d}{dt} f(p \exp(tX)g) \mid_{t=0} \\ &= \frac{d}{dt} f(pgg^{-1} \exp(tX)g) \mid_{t=0} \\ &= \frac{d}{dt} f(pg \exp(tAd_{g^{-1}}X)) \mid_{t=0} \\ &= (Ad_{g^{-1}}X)^*_{pg}(f). \end{aligned}$$

□

Proposition 2.1.7. *Let $\{U_a, \psi_a\}$ be a trivialization of the principal bundle $\pi : P \to M$. For $\psi_a = (\pi, g_a)$, we have*

$$(g_a)_*(X^*_p) = (L_{g_a(p)})_*(X).$$

Proof. Taking $f \in C^\infty(G)$, we get

$$\begin{aligned}(g_a)_*(X_p^*)(f) = X_p^*(f \circ g_a) = (\sigma_p)_{*,e}(X)(f \circ g_a) \\ = X(f \circ g_a \circ \sigma_p) \\ = X(f \circ L_{g_a(p)}) \\ = (L_{g_a(p)})_*(X)(f).\end{aligned}$$

□

Given a principal G-bundle $\pi : P \to M$ and an element m in M, on each fiber T_pP, where $p \in \pi^{-1}(m)$, we define the *vertical subspace* V_p by

$$V_p = \ker(\pi_{*,p}) : T_pP \to T_mM.$$

We call $X \in \mathfrak{X}(P)$ a *vertical vector field* if $X(p) \in V_p$ for all $p \in P$.

In general, if X, Y are vertical vector fields, so is $[X, Y]$. Indeed, if we have $f \in C^\infty(P)$, we obtain

$$\begin{aligned}\pi_*([X, Y]_p)(f) = [X, Y]_p(f \circ \pi) = X_p(Y(f \circ \pi)) - Y_p(X(f \circ \pi)) \\ = X_p(\pi_*(Y)(f)) - Y_p(\pi_*(X)(f),\end{aligned}$$

and this implies $\pi_*([X, Y]_p) = 0$.

Furthermore, V_p defines a G-invariant distribution V on TP. Indeed, it is sufficient to show that $(R_g)_*(V_p) = V_{pg}$ holds for all $g \in G$. Since we have $\pi \circ (R_g) = \pi$, for $u \in V_p$, we obtain

$$\pi_*((R_g)_*(u)) = \pi_*(u) = 0.$$

Thus, we get $(R_g)_*(u) \in V_{pg}$.

Moreover, every fundamental vector field is a vertical vector field. We can prove this as follows. Let X be in $\mathfrak{g}$ and X^* be its associated fundamental vector field. We will show that $X_p^* \in \ker(\pi_*)$ holds for all $p \in P$. Fixing $f \in C^\infty(P)$, we obtain

$$\begin{aligned}\pi_*(X_p^*)(f) = X_p^*(f \circ \pi) = \frac{d}{dt}(p\exp(tX))\,|_{t=0}\,(f \circ \pi) \\ = \frac{d}{dt}(\pi(p\exp(tX)))\,|_{t=0}\,(f) \\ = \frac{d}{dt}(\pi(p))\,|_{t=0}\,(f).\end{aligned}$$

Since $\frac{d}{dt}(\pi(p))\mid_{t=0}(f) = 0$, we conclude $X^*_p \in V_p$. Finally, since $\dim(V_p) = \dim(\mathfrak{g})$, we get $V_p \cong \mathfrak{g}$.

Definition 2.1.8. A *connection* on P is a smooth choice of *horizontal subspaces* H_p on each fiber T_pP that satisfies

i. $T_pP = H_p \oplus V_p$,

ii. $(R_g)_*(H_p) = H_{pg}$.

Each connection H on P has associated a $\mathfrak{g}$-valued 1-form ω, defined as follows.

For $p \in \pi^{-1}(m) \subset P$, we have $T_pP = H_p \oplus V_p$. If we take $u \in T_pP$, we can express it uniquely as $u = u_h + u_v$ where $u_h \in H_p$ and $u_v \in V_p$. Bear in mind that u_v can be seen as X^*_p, where X^* is the fundamental vector field induced by some unique $X \in \mathfrak{g}$. Then the $\mathfrak{g}$-valued 1-form ω is taken to be $\omega(u) = X$.

This $\mathfrak{g}$-valued 1-form ω on P is called the *connection form* of H. Notice the equality $\ker(\omega_p) = H_p$ for all $p \in P$.

Proposition 2.1.9. *The connection form ω verifies the equality*

$$(R_g)^*\omega = Ad_{g^{-1}} \circ \omega.$$

Proof. Take $u = u_h + u_v \in T_pP$, where $u_h \in H_p$ and $u_v \in V_p$. Suppose $X = \omega_p(u)$. From Proposition 2.1.6 we get

$$\begin{aligned}(R_g)^*\omega_p(u) = \omega_{pg}((R_g)_*(u)) &= \omega_{pg}((R_g)_*u_h) + \omega_{pg}((R_g)_*u_v)\\ &= \omega_{pg}((R_g)_*X^*_p)\\ &= \omega_{pg}((Ad_{g^{-1}}X)^*_{pg})\\ &= Ad_{g^{-1}}X\\ &= Ad_{g^{-1}}(\omega_p(u)).\end{aligned}$$

□

Remark 2.1.10. Whenever we have a $\mathfrak{g}$-valued 1-form ω that satisfies

i. $\omega_p(u) = X$, where $X^*_p = u_v$, and

ii. $(R_g)^*\omega = Ad_{g^{-1}} \circ \omega$, then this ω will provide a connection H on a principal G-bundle $\pi : P \to M$. The horizontal subspaces H_p are defined simply by $H_p = \ker(\omega_p)$.

Next we study the local behavior of the connection form on a principal bundle. Let ω be the connection form on P associated to a connection $H \subset TP$. For the local trivialization (U_α, ψ_α), where $\psi_\alpha = (\pi, g_\alpha)$, we have associated to these trivializations a collection of canonical sections s_α in such a way that $s_\alpha(m) = \psi_\alpha^{-1}(m, e)$. These sections allow us to define $\mathfrak{g}$-valued 1-forms on U_α via

$$A_\alpha = s_\alpha^* \omega.$$

Proposition 2.1.11. *Let ω be the connection form on P. Suppose (U_α, ψ_α) is a local trivialization of this principal bundle P. If we write $\omega_\alpha = \omega|_{\pi^{-1}(U_\alpha)}$, then on each $\pi^{-1}(U_\alpha)$, we have the equality*

$$\omega_\alpha = Ad_{g_\alpha^{-1}} \circ \pi^* A_\alpha + g_\alpha^* \theta.$$

Proof. First we prove this equality on $Im(s_\alpha)$. We can express T_pP as

$$T_pP = Im((s_\alpha)_* \circ \pi_*) \oplus V_p,$$

for all $p \in Im(s_\alpha)$. Indeed, if we take $u \in T_pP$, then we have

$$\begin{aligned} \pi_*(u - (s_\alpha)_* \circ \pi_*(u)) &= \pi_*(u) - (\pi \circ s_\alpha)_*(\pi_*(u)) \\ &= \pi_*(u) - \pi_*(u) \\ &= 0. \end{aligned}$$

Thus, we get $u - (s_\alpha)_* \circ \pi_*(u) \in V_p$, and so, $T_pP = Im((s_\alpha)_*\pi_*) + V_p$. In addition, we need to verify $Im((s_\alpha)_* \circ \pi_*) \cap V_p = \{0\}$. Consider $u \in Im((s_\alpha)_* \circ \pi_*) \cap V_p$, so that $u = (s_\alpha)_* \circ \pi_*(v)$ for some $v \in T_pP$ and $\pi_*(u) = 0$. In this case we reach

$$0 = \pi_*(u) = \pi_*((s_\alpha)_* \circ \pi_*(v)) = \pi_*(v),$$

which implies $u = 0$.

To establish the proposition, we take $u \in T_pP$ with $p \in Im(s_\alpha)$, which can be written as

$$u = (s_\alpha)_* \circ \pi_*(u) + u_v,$$

where $u_v \in V_p$. Since $g_a(p) = e$, we have

$$Ad_{g_\alpha^{-1}(p)} \circ \pi^* A_a(u) = A_a(\pi_*(u)) = \omega((s_a)_* \pi_*(u)) = \omega(u - u_v).$$

On the other hand we have

$$\begin{aligned} g_a^* \theta(u) = \theta_e((g_a)_*(u)) &= (g_a)_*(u) \\ &= (g_a)_*((s_a)_* \circ \pi_*(u) + u_v) \\ &= (g_a \circ s_a)_*(\pi_*(u)) + (g_a)_*(u_v). \end{aligned}$$

As $(g_a \circ s_a)(m) = e$ for all $m \in M$, we obtain $(g_a \circ s_a)_*(\pi_*(u)) = 0$. Therefore, we have $g_a^* \theta(u) = (g_a)_*(u_v)$. Let $X = \omega(u_v)$. Then, by Proposition 2.1.7, we get

$$(g_a)_*(u_v) = (L_{g_\alpha(p)})_*(X) = X = \omega(u_v),$$

and thus

$$Ad_{g_\alpha^{-1}(p)} \circ \pi^* A_a(u) + g_a^* \theta(u) = \omega(u - u_v) + \omega(u_v) = \omega(u).$$

For the general case, taking $pg \in P$ and $u \in T_{pg}P$, where $p \in Im(s_a)$, we find

$$\begin{aligned} Ad_{g_\alpha^{-1}(pg)} \circ \pi^* A_a \mid_{pg} (u) &= Ad_{g_\alpha^{-1}(pg)} \circ \pi^* A_a \mid_{pg} ((R_g)_{*,p}(u_0)) \\ &= Ad_{g_\alpha^{-1}(pg)} \circ A_a(\pi_{*,pg} \circ (R_g)_{*,p}(u_0)) \\ &= Ad_{g_\alpha^{-1}(pg)} \circ A_a(\pi_{*,p}(u_0)) \\ &= Ad_{g^{-1}} \circ Ad_{g_\alpha^{-1}(p)} \circ \pi^* A_a(u_0), \end{aligned}$$

where $u_0 \in T_pP$ and $u = (R_g)_{*,p}(u_0)$; while for the other component we have

$$g_a^* \theta \mid_{pg} (u) = g_a^* \theta \mid_{pg} ((R_g)_{*,p}(u_0)) = (R_g)^* g_a^* \theta \mid_p (u_0) = (g_a \circ R_g)^* \theta(u_0).$$

Since $g_a \circ R_g = R_g \circ g_a$, this results in

$$(g_a \circ R_g)^* \theta(u_0) = (R_g \circ g_a)^* \theta(u_0) = g_a^*(R_g^* \theta)(u_0).$$

As $R_g^* \theta = Ad_{g^{-1}} \circ \theta$, we get

$$\begin{aligned} g_a^*(R_g^* \theta)(u_0) &= g_a^*(Ad_{g^{-1}} \circ \theta)(u_0) \\ &= Ad_{g^{-1}} \circ \theta((g_a)_*(u_0)) \\ &= Ad_{g^{-1}} \circ (g_a)^* \theta(u_0). \end{aligned}$$

At the end we arrive to

$$Ad_{g_\alpha^{-1}(pg)} \circ \pi^* A_\alpha |_{pg} (u) + g_\alpha^* \theta |_{pg} (u) = Ad_{g^{-1}} (Ad_{g_\alpha^{-1}(p)} \circ \pi^* A_\alpha(u_0) + (g_\alpha)^* \theta(u_0))$$
$$= Ad_{g^{-1}} (\omega_\alpha |_p (u_0))$$
$$= \omega_\alpha((R_g)_{*,p}(u_0))$$
$$= \omega_\alpha(u).$$

□

Take (U_α, ψ_α) and (U_β, ψ_β) as two local trivializations of a principal bundle $\pi : P \to M$. Let s_α and s_β be corresponding canonical sections that define the $\mathfrak{g}$-valued 1-forms A_α and A_β. On the intersection $U_\alpha \cap U_\beta$, the following relation between A_α and A_β holds:

$$A_\alpha = s_\alpha^* \omega_\alpha = s_\alpha^* \omega_\beta = s_\alpha^* (Ad_{g_\beta(s_\alpha)^{-1}} \circ \pi^* A_\beta + g_\beta^* \theta)$$
$$= s_\alpha^* (Ad_{g_\beta(s_\alpha)^{-1}} \circ \pi^* A_\beta) + s_\alpha^* (g_\beta^* \theta)$$
$$= s_\alpha^* (Ad_{g_\beta(s_\alpha)^{-1}} \circ \pi^* A_\beta) + (g_\beta \circ s_\alpha)^* \theta$$
$$= s_\alpha^* (Ad_{g_{\alpha\beta}} \circ \pi^* A_\beta) + (g_{\beta\alpha})^* \theta$$
$$= Ad_{g_{\alpha\beta}} \circ (s_\alpha^* \pi^* A_\beta) + (g_{\beta\alpha})^* \theta$$
$$= Ad_{g_{\alpha\beta}} \circ A_\beta + (g_{\beta\alpha})^* \theta.$$

Equivalently, from

$$A_\alpha = Ad_{g_{\alpha\beta}} \circ A_\beta + (g_{\beta\alpha})^* \theta,$$

we obtain

$$Ad_{g_{\beta\alpha}} \circ A_\alpha = Ad_{g_{\beta\alpha}} (Ad_{g_{\alpha\beta}} \circ A_\beta) + Ad_{g_{\beta\alpha}} \circ (g_{\beta\alpha})^* \theta$$
$$= A_\beta + Ad_{g_{\beta\alpha}} \circ (g_{\beta\alpha})^* \theta.$$

Thus, we get

$$A_\beta = Ad_{g_{\beta\alpha}} \circ A_\alpha - Ad_{g_{\beta\alpha}} \circ (g_{\beta\alpha})^* \theta.$$

Interchanging α and β, this results in

$$A_\alpha = Ad_{g_{\alpha\beta}} \circ A_\beta - Ad_{g_{\alpha\beta}} \circ (g_{\alpha\beta})^* \theta$$
$$= Ad_{g_{\alpha\beta}} \circ (A_\beta - (g_{\alpha\beta})^* \theta).$$

For example, if G is a matrix group, we obtain

$$A_\alpha = g_{\alpha\beta} A_\beta g_{\alpha\beta}^{-1} - dg_{\alpha\beta}.g_{\alpha\beta}^{-1},$$

where d represents the exterior derivative in each entry of the matrix $g_{\alpha\beta}$.

Remark 2.1.12. Reciprocally, given a family of local 1-forms $A_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$ that satisfies the above equations, we can define a local 1-form $\omega_\alpha \in \Omega^1(P) \otimes \mathfrak{g}$ by

$$\omega_\alpha = Ad_{g_\alpha^{-1}} \circ \pi^* A_\alpha + g_\alpha^* \theta.$$

This allows us to glue them into a global 1-form $\omega = \omega_\alpha$ (on each $\pi^{-1}(U_\alpha)$). This 1-form ω happens to be a connection form on P.

Remark 2.1.13. When we have two connection forms ω and $\tilde{\omega}$, their difference $\omega - \tilde{\omega}$ is a *horizontal* 1-form on P. This means that $(\omega - \tilde{\omega})(v) = 0$ holds for all vertical vectors v.

Let H be a connection on a principal bundle $\pi : P \to M$. The *horizontal projection* on each fiber T_pP is defined by

$$\begin{aligned} h_p : \quad & T_pP \longrightarrow H_p \\ & u = u_h + u_v \longrightarrow h_p(u) = u_h. \end{aligned}$$

Notice here that $\ker(h_p) = V_p$ and $Im(h_p) = H_p$ for each $p \in P$. Furthermore, we get $h \circ (R_g)_* = (R_g)_* \circ h$: indeed, fixing $u \in T_pP$, we have

$$h \circ (R_g)_*(u) = h((R_g)_*(u_h) + (R_g)_*(u_v)) = (R_g)_*(u_h) = (R_g)_* h(u).$$

For $\alpha \in \Omega^1(P)$, we write $h^*\alpha = \alpha \circ h$. In general, for $\beta \in \Omega^k(P)$, the form $h^*\beta$ is defined by

$$h^*\beta(u_1, ..., u_k) = \beta(h(u_1), ...h(u_k)).$$

The *curvature form* Ω of a connection H is defined as the $\mathfrak{g}$-valued 2-form $\Omega = h^* d\omega$, here ω is the connection form of H. From the definition of curvature we obtain

$$\begin{aligned} \Omega(u, v) = h^* d\omega(u, v) &= d\omega(h(u), h(v)) \\ &= d\omega(u_h, v_h) \\ &= u_h(\omega(v_h)) - v_h(\omega(u_h)) - \omega([u_h, v_h]) \\ &= -\omega([u_h, v_h]). \end{aligned}$$

Observe that if the distribution were integrable, then Ω would vanish. The converse also is true.

Proposition 2.1.14. ***[Structure Equation]*** *Let ω the connection form of the connection $H \subset TP$, and Ω the corresponding curvature form. Then we have*

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega].$$

Proof. Let X, Y be vector fields over P. We are going to prove this theorem for diverse X and Y. Since $\Omega(X, Y) = d\omega(hX, hY)$, it is sufficient to show the equality

$$\begin{aligned} d\omega(hX, hY) &= d\omega(X, Y) + \frac{1}{2}[\omega, \omega](X, Y) \\ &= d\omega(X, Y) + \frac{1}{2}[\omega(X), \omega(Y)]. \end{aligned}$$

If X and Y were horizontal vector fields, then $X = hX$ and $Y = hY$. Thus, $\omega(X) = \omega(Y) = 0$, and we get $d\omega(hX, hY) = d\omega(X, Y)$.

If X and Y were vertical vector fields, then on some neighbourhood U of each point $p \in P$ we can choose fundamental vector fields X^* and Y^* such that $X = X^*$ and $Y = Y^*$. Suppose $\tilde{X}$ and $\tilde{Y}$ in $\mathfrak{g}$ induce the fundamental vector fields X^* and Y^*, respectively. If we notice the equality $d\omega(hX, hY) = d\omega(0,0) = 0$, we reach

$$\begin{aligned} d\omega(X^*, Y^*) + \frac{1}{2}[\omega(X^*), \omega(Y^*)] &= \frac{1}{2}\{X^*(\omega(Y^*)) - Y^*(\omega(X^*)) - \omega([X^*, Y^*])\} \\ &\quad + \frac{1}{2}[\omega(X^*), \omega(Y^*)] \\ &= \frac{1}{2}\{X^*(\tilde{Y}) - Y^*(\tilde{X}) - \omega([X^*, Y^*])\} + \frac{1}{2}[\tilde{X}, \tilde{Y}] \\ &= -\frac{1}{2}\omega([X^*, Y^*]) + \frac{1}{2}[\tilde{X}, \tilde{Y}] \\ &= -\frac{1}{2}\omega([\sigma(\tilde{X}), \sigma(\tilde{Y})]) + \frac{1}{2}[\tilde{X}, \tilde{Y}] \\ &= -\frac{1}{2}\omega(\sigma([\tilde{X}, \tilde{Y}])) + \frac{1}{2}[\tilde{X}, \tilde{Y}] \\ &= -\frac{1}{2}[\tilde{X}, \tilde{Y}] + \frac{1}{2}[\tilde{X}, \tilde{Y}] \\ &= 0. \end{aligned}$$

Thus, the equation is also verified for this case.

If X is horizontal and Y is vertical, we can choose as above a fundamental vector field Y^* and an element $\tilde{Y} \in \mathfrak{g}$ that induces Y^*, so that $Y = Y^*$ on a neighbourhood of $p \in P$. As above, we have

$$\Omega(X, Y) = d\omega(hX, hY) = d\omega(X, 0) = 0,$$

then we require

$$d\omega(X, Y^*) + \frac{1}{2}[\omega(X), \omega(Y^*)] = 0.$$

Since X is horizontal, we get $\omega(X) = 0$. Thus, the above equality is equivalent to

$$\begin{aligned} 0 = d\omega(X, Y^*) &= \frac{1}{2}\{X(\omega(Y^*)) - Y^*(\omega(X)) - \omega([X, Y^*])\} \\ &= -\frac{1}{2}\omega([X, Y^*]). \end{aligned}$$

This means that we must prove that $[X, Y^*]$ is horizontal. For this, we remember that if X and Y are vector fields on P, and $\{\Phi_t\}$ is the 1-parameter group associated to X, then we have

$$[X, Y] = \lim_{t \to 0} \frac{1}{t}[Y - (\Phi_t)_* Y].$$

Thus, for Y^* and X we get

$$[Y^*, X] = \lim_{t \to 0} \frac{1}{t}[X - (R_{\exp(tY)})_* X].$$

Since X and $(R_{\exp(tY)})_* X$ are horizontal, we conclude $[Y^*, X]$ is horizontal. □

Since S^1 is an abelian group, the structure equation for principal S^1-bundles is simply

$$\Omega = d\omega.$$

Next we study the local behaviour of the curvature form Ω on principal S^1-bundles.

If we consider canonical sections $s_a : U_a \to P$ and we pull back Ω via said sections, we obtain locally $\mathfrak{g}$-valued 2-forms defined on U_a

$$F_a = s_a^* \Omega \in \Omega^2(U_a) \otimes \mathfrak{g}.$$

Then, using the structure equation, we obtain

$$F_a = s_a^* \Omega = s_a^*(d\omega + \frac{1}{2}[\omega, \omega]) = d(s_a^*\omega) + \frac{1}{2}s_a^*[\omega, \omega].$$

By the linearity of the product on $\Omega^2(U_a) \otimes \mathfrak{g}$, it follows

$$\begin{aligned}
F_a &= dA_a + \frac{1}{2}s_a^*(\omega^i \wedge \omega^j \otimes [g_i, g_j]) \\
&= dA_a + \frac{1}{2}(s_a^*\omega^i \wedge s_a^*\omega^j) \otimes [g_i, g_j] \\
&= dA_a + \frac{1}{2}(A_a^i \wedge A_a^j) \otimes [g_i, g_j] \\
&= dA_a + \frac{1}{2}[A_a, A_a].
\end{aligned}$$

Now we are going to study what happens in the intersections $U_{\alpha\beta}$. Recall that on $U_{\alpha\beta}$ we have $A_a = Ad_{g_{\alpha\beta}} \circ A_\beta + g_{\beta a}^*\theta$. We then get

$$\begin{aligned}
F_a &= dA_a + \frac{1}{2}[A_a, A_a] \\
&= d(Ad_{g_{\alpha\beta}} \circ A_\beta + g_{\beta a}^*\theta) + \frac{1}{2}[Ad_{g_{\alpha\beta}} \circ A_\beta + g_{\beta a}^*\theta, Ad_{g_{\alpha\beta}} \circ A_\beta + g_{\beta a}^*\theta] \\
&= d(Ad_{g_{\alpha\beta}} \circ A_\beta) + d(g_{\beta a}^*\theta) + \frac{1}{2}[Ad_{g_{\alpha\beta}} \circ A_\beta, Ad_{g_{\alpha\beta}} \circ A_\beta] + \\
&\quad \frac{1}{2}[Ad_{g_{\alpha\beta}} \circ A_\beta, g_{\beta a}^*\theta] + \frac{1}{2}[g_{\beta a}^*, Ad_{g_{\alpha\beta}} \circ A_\beta] + \frac{1}{2}[g_{\beta a}^*\theta, g_{\beta a}^*\theta].
\end{aligned}$$

Since $d(g_{\beta a}^*\theta) = g_{\beta a}^*(d\theta)$ and $[Ad_{g_{\alpha\beta}} \circ A_\beta, g_{\beta a}^*\theta] + [g_{\beta a}^*, Ad_{g_{\alpha\beta}} \circ A_\beta] = 0$ are satisfied, with the Maurer-Cartan structure equation we obtain

$$\begin{aligned}
F_a &= d(Ad_{g_{\alpha\beta}} \circ A_\beta) + g_{\beta a}^*(d\theta) + \frac{1}{2}[Ad_{g_{\alpha\beta}} \circ A_\beta, Ad_{g_{\alpha\beta}} \circ A_\beta] + \frac{1}{2}[g_{\beta a}^*\theta, g_{\beta a}^*\theta] \\
&= d(Ad_{g_{\alpha\beta}} \circ A_\beta) + g_{\beta a}^*(-\frac{1}{2}[\theta, \theta]) + \frac{1}{2}[Ad_{g_{\alpha\beta}} \circ A_\beta, Ad_{g_{\alpha\beta}} \circ A_\beta] + \frac{1}{2}[g_{\beta a}^*\theta, g_{\beta a}^*\theta] \\
&= d(Ad_{g_{\alpha\beta}} \circ A_\beta) - \frac{1}{2}[g_{\beta a}^*\theta, g_{\beta a}^*\theta] + \frac{1}{2}[Ad_{g_{\alpha\beta}} \circ A_\beta, Ad_{g_{\alpha\beta}} \circ A_\beta] + \frac{1}{2}[g_{\beta a}^*\theta, g_{\beta a}^*\theta] \\
&= d(Ad_{g_{\alpha\beta}} \circ A_\beta) + \frac{1}{2}[Ad_{g_{\alpha\beta}} \circ A_\beta, Ad_{g_{\alpha\beta}} \circ A_\beta] \\
&= Ad_{g_{\alpha\beta}} \circ dA_\beta + \frac{1}{2}Ad_{g_{\alpha\beta}} \circ [A_\beta, A_\beta] \\
&= Ad_{g_{\alpha\beta}} \circ (dA_\beta + \frac{1}{2}[A_\beta, A_\beta]) \\
&= Ad_{g_{\alpha\beta}} \circ F_\beta.
\end{aligned}$$

Thanks to the above calculation we can establish the following result.

Proposition 2.1.15. *If G is an Abelian group, then $F_a = s_a^*\Omega$ defines a global 2-form $F \in \Omega^2(M) \otimes \mathfrak{g}$ by $F_a = s_a^*\Omega$.* □

We conclude this section with some results concerning the existence of forms on M when $G = S^1$. In this case we have $\mathfrak{g} = \mathbb{R}$.

Proposition 2.1.16. *Let Ω be a curvature form on P. Then there exists a unique 2-form Γ on M such that*

$$\Omega = \pi^*\Gamma.$$

Furthermore, considering the (real) coordinates given by the composition of the trivialization $\psi_a : \pi^{-1}(U_a) \to U_a \times S^1$ and the chart $\varphi_a = (x_1, x_2, ..., x_n)$:
$U_a \to \mathbb{R}^n$, we can express the 2-form Γ as

$$\Gamma = \sum_{i,j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_i \wedge dx_j,$$

where these functions f_i depend only on $x_1, ..., x_n$.

Proof. See Morita [22]. □

Proposition 2.1.17. *If ω and $\tilde{\omega}$ are two connection forms on P, then there is a 1-form τ on M such that*

$$\omega - \tilde{\omega} = \pi^*\tau.$$

Conversely, for any $\tau \in \Omega^1(M)$ and ω a connection form on P, it satisfies that $\omega + \pi^*\tau$ is a connection form on P.

Proof. See Morita [22]. □

2.2 Connections on linear frame bundles

Let M be a n-dimensional smooth manifold. For each $m \in M$ we define the set

$$L_m(M) = \{(m, \xi^1, ..., \xi^n) : \{\xi^1, ..., \xi^n\} \text{ is a basis of } T_mM\}.$$

The union of all $L_m(M)$, with $m \in M$, is called the *linear frame bundle* of M and is denoted by $L(M)$. The set $L(M)$ can be endowed with the structure of a smooth manifold as follows. Locally, $L(M)$ is the product of a neighbourhood $U \subset M$ with the linear group $GL_n(\mathbb{R})$. For that we take first a chart (U, φ) around $m \in M$ and next we choose a local frame $\{X^1, ..., X^n\}$

on U. Each basis $\{\xi^1, ..., \xi^n\}$ of T_mM is uniquely determined by a matrix $A_\xi = (A^i{}_j) \in GL_n(\mathbb{R})$ since we can express it as

$$\xi^j = A^i_j X^i(m).$$

Thus, we can establish a bijective map

$$\begin{aligned} \Phi: \quad & L_m(M) \longrightarrow \varphi(U) \times \mathbb{R}^{n\times n} \\ & \xi = (m, \xi^1, ..., \xi^n) \longrightarrow \Phi(\xi) = (\varphi(m), A_\xi), \end{aligned}$$

where m takes values on U. These maps Φ endows $L(M)$ of a smooth structure. Since $L(M)$ looks locally as $U \times GL_n(\mathbb{R})$, it is natural to impose the structure of a principal $GL_n(\mathbb{R})$-bundle on $L(M)$. We define the projection map $\pi : L(M) \to M$ as $\pi(m, \xi^1, ..., \xi^n) = m$. Clearly we have $\pi^{-1}(m) \cong GL_n(\mathbb{R})$. The action of $GL_n(\mathbb{R})$ on $L(M)$ is given by

$$\xi A = (m, \sum_j A^1_j \xi^1, ..., \sum_j A^n_j \xi^n),$$

where $\xi = (m, \xi^1, ..., \xi^n)$ and $A = (A^i_j)$. Obviously we have $(\xi A)B = \xi(AB)$, $\xi I_n = \xi$, with the free action.

To define a connection on $L(M)$ we begin by defining the vertical subspace V^ξ on each fiber $T_\xi L(M)$ as $V_\xi = \ker(\pi_{*,\xi})$. The connection H on $L(M)$ is determined by the choice of a horizontal subspaces H_ξ on each fiber subject to

(i) $T_\xi L(M) = V_\xi \oplus H_\xi$,

(ii) $(R_A)_*(H_\xi) = H_{\xi A}$, for all $A \in GL_n(\mathbb{R})$.

Next we study the connection form induced by H. First, we will consider the fundamental vector fields on $L(M)$. These fundamental vector fields are uniquely determined by $\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$, the Lie algebra of $GL_n(\mathbb{R})$. Due to this isomorphism, we look for elements of the basis $\{E^i_j\}$ of $M_n(\mathbb{R})$, where $E^i{}_j$ represents the $n \times n$ matrix with all its entries 0 but the (i, j)- position which equals 1. For each $\xi = (m, \xi^1, ..., \xi^n) \in L_m(M)$ we define

$$\begin{aligned} \sigma_\xi: \quad & GL_n(\mathbb{R}) \longrightarrow L(M) \\ & A \longrightarrow \sigma_\xi(A) = \xi A. \end{aligned}$$

Then the basis $\{X^i_j\}$ for the set of fundamental vector fields associated to $\{E^i_j\}$ is obtained as

$$X^i_j(\xi) = (\sigma_\xi)_{*,I_n}(E^i_j).$$

We can also interpret the above construction as follows. Since the vector E^i_j on $T_{I_n}GL_n(\mathbb{R}) \cong M_n(\mathbb{R})$ can be obtained as the derivative in 0 of the curve $\gamma(t) = I_n + tE^i_j$, we have $E^i_j = \dot{\gamma}(0)$. Thus we get

$$X^i_j(\xi) = (\sigma_\xi)_{*,I_n}(E^i_j) = (\sigma_\xi)_{*,I_n}(\dot{\gamma}(0)) = \dot{\overbrace{\sigma_\xi \circ \gamma}}(0),$$

where $\sigma_\xi \circ \gamma(t) = \xi(I_n + tE^i_j)$.

Given the basis $\{X^i_j\}$ for the space of fundamental vector fields on $L(M)$, we write its dual basis as $\{\varphi^j_i\}$. The connection form induced by H is then the $\mathfrak{gl}_n(\mathbb{R})$-valued form $\varphi = (\varphi^i_j)$.

Proposition 2.2.1. *A connection H on $L(M)$ induces a covariant derivative on M.*

Proof. We consider a local section frame $\{\xi^1, ..., \xi^n\}$ on $U \subset M$. This local basis can be understood as a local section ξ on $L(M)$ given by

$$\begin{aligned} \xi : \quad U &\longrightarrow L(M) \\ m &\longrightarrow \xi(m) = (m, \xi^1(m), ..., \xi^n(m)). \end{aligned}$$

Let $\varphi = (\varphi^i_j)$ be the connection form of H, whose pullback $\xi^*\varphi$ is a $\mathfrak{gl}_n(\mathbb{R})$-valued form on U. The matrix $\xi^*\varphi$ will describe the connection form of a covariant derivative on M. Writing $\xi^*\varphi$ as a matrix $\omega = (\omega^i_j)$ with ω^i_j 1 forms, the covariant derivative ∇ on U is given by

$$\nabla_X Y = \sum_j \nabla_X Y_j \xi^j = \sum_j X(Y_j)\xi^j + Y_j \nabla_X \xi^j,$$

where $\nabla_X \xi^j = \omega^j_i(X)\xi^i$. It is clear that ∇ satisfies the Leibniz rule and verifies $\nabla_{fX}Y = f\nabla_X Y$ for all $f \in C^\infty(U)$. Thus, we can take an open covering $\{U_a\}$ of M and a collection of local sections $\xi_a : U_a \to L(M)$. Next we define on each U_a the connection ∇^a, where its connection form is given by the $\mathfrak{gl}_n(\mathbb{R})$-valued form $\xi^*_a\varphi$. If $\{\sigma_a\}$ is a locally finite partition of the unity for the covering $\{U_a\}$, then we define the covariant derivative ∇ globally on M by

$$\nabla_X Y = \sum_a \sigma_a \nabla^a_X Y.$$

$\square$

2.3 The group structure of the set of principal S^1-bundles

We denote by $\mathsf{P}(M, S^1)$ the set of all principal S^1-bundles on M. As we will see next, $\mathsf{P}(M, S^1)$ can be endowed with a group structure.

For two elements $\pi : P \to M$ and $\tilde{\pi} : \tilde{P} \to M$ in $\mathsf{P}(M, S^1)$, we define the set

$$\Delta(P \times \tilde{P}) = \{(u, \tilde{u}) \in P \times \tilde{P} : \pi(u) = \tilde{\pi}(\tilde{u})\}.$$

On $\Delta(P \times \tilde{P})$, we define a action of S^1 by $(u, \tilde{u}) \cdot s = (us, \tilde{u}s)$. This action establishes a equivalence relation $\sim$, which is given by $(u, \tilde{u}) \sim (v, \tilde{v})$ whenever there exists $s \in S^1$ such that $v = us$ and $\tilde{v} = \tilde{u}s^{-1}$. Since S^1 is compact and the action is smooth and free, we can say that $\Delta(P \times \tilde{P})/\sim$ is a smooth manifold.

We define a group operation on $\mathsf{P}(M, S^1)$. For this, take P and $\tilde{P}$ in $\mathsf{P}(M, S^1)$. We define the element $P + \tilde{P} \in \mathsf{P}(M, S^1)$ as the quotient space $\Delta(P \times \tilde{P})/\sim$.

The S^1-action on $P + \tilde{P}$ is given by $[(u, \tilde{u})]s = [(us, \tilde{u})]$, with $s \in S^1$. This S^1-action is well defined because if $(u, \tilde{u}) \sim (v, \tilde{v})$, then there is $t \in S^1$ such that $v = ut$ and $\tilde{v} = \tilde{u}t^{-1}$. Then $vs = us(t)$, which implies $(us, \tilde{u}) \sim (vs, \tilde{v})$.

On the other hand, S^1 acts freely on $P + \tilde{P}$. This is because if $[(u, \tilde{u})]s = [(u, \tilde{u})]$ for all $[(u, \tilde{u})] \in P + \tilde{P}$, then $(us, \tilde{u}) \sim (u, \tilde{u})$. Since the action on the first component is free, we have that $s = 1$. Next we define the projection map $\pi^{\jmath}$ as

$$\begin{array}{rccl} \pi^{\jmath}: & P + \tilde{P} & \longrightarrow & M \\ & [(u, \tilde{u})] & \longrightarrow & \pi^{\jmath}([(u, \tilde{u})]) = \pi(u). \end{array}$$

Let us see that S^1 acts transitively on each fiber $(\pi^{\jmath})^{-1}(m)$, where $m \in M$. Taking $[(u, \tilde{u})]$ and $[(v, \tilde{v})]$ in $(\pi^{\jmath})^{-1}(m)$, we have $\pi(u) = \pi(v)$ on M. Then there exists $s \in S^1$ such that $v = us$. Therefore, $[(v, \tilde{v})] = [(u, \tilde{u})]s$.

Now we verify that $P + \tilde{P}$ is locally trivial. Indeed, consider the trivializations $\{(U_\alpha, \psi_\alpha)\}$, $\{(\tilde{U}_\beta, \psi_\beta)\}$ of P and $\tilde{P}$, respectively. We define the open covering $\{U^{\jmath}_{\alpha\beta}\}$ of M, where $U^{\jmath}_{\alpha\beta} = U_\alpha \cap \tilde{U}_\beta$ whenever $U_\alpha \cap \tilde{U}_\beta \neq \emptyset$. On $(\pi^{\jmath})^{-1}(U^{\jmath}_{\alpha\beta})$ we define the map $\psi^{\jmath}_{\alpha\beta}$ as

$$\psi^{\jmath}_{\alpha\beta}([(u, \tilde{u})]) = (\pi^{\jmath}([(u, \tilde{u})]), g^{\jmath}_{\alpha\beta}([(u, \tilde{u})])),$$

where $g^{j}_{\alpha\beta}([(u,\tilde{u})]) = g_\alpha(u)\tilde{g}_\beta(\tilde{u})$, $\psi_\alpha = (\pi, g_\alpha)$ and $\tilde{\psi}_\beta = (\tilde{\pi}, \tilde{g}_\beta)$. Let us see that $\psi^{j}_{\alpha\beta}$ is a diffeomorphism on $(\pi^{j})^{-1}(U^{j}_{\alpha\beta})$. Clearly it is smooth. For the injectivity of $\psi^{j}_{\alpha\beta}$, we take two elements $[(u,\tilde{u})], [(v,\tilde{v})] \in P + \tilde{P}$, such that $\psi^{j}_{\alpha\beta}([(u,\tilde{u})]) = \psi^{j}_{\alpha\beta}([(v,\tilde{v})])$. Then we have $(\pi^{j}([(u,\tilde{u})]), g^{j}_{\alpha\beta}([(u,\tilde{u})])) = (\pi^{j}([(v,\tilde{v})]), g^{j}_{\alpha\beta}([(v,\tilde{v})]))$. By the above equality we have $\pi^{j}([(u,\tilde{u})]) = \pi^{j}([(v,\tilde{v})])$ and $g^{j}_{\alpha\beta}([(u,\tilde{u})]) = g^{j}_{\alpha\beta}([(v,\tilde{v})])$. These last equalities are equivalent to $\pi(u) = \pi(v)$ and $g_\alpha(u)g_\beta(\tilde{u}) = g_\alpha(v)g_\beta(\tilde{v})$, respectively. As $\pi(u) = \pi(v)$, there exists $t \in S^1$ such that $v = ut$. Then we have

$$\begin{aligned} g_\beta(\tilde{u}) &= (g_\alpha(u))^{-1}g_\alpha(v)g_\beta(\tilde{v}) \\ &= (g_\alpha(u))^{-1}(g_\alpha(u)t)g_\beta(\tilde{v}) \\ &= g_\beta(\tilde{v})t \\ &= g_\beta(\tilde{v}t). \end{aligned}$$

Thus, $\tilde{u} = \tilde{v}t$, which implies $(u, \tilde{u}) \sim (v, \tilde{v})$. As a consequence $\psi^{j}_{\alpha\beta}$ is injective. The surjectivity is trivial.

Finally, let us see why $g^{j}_{\alpha\beta}$ is S^1-equivariant. In fact, taking $[(u, \tilde{u})] \in P + \tilde{P}$ and $s \in S^1$, we have

$$g^{j}_{\alpha\beta}([(u,\tilde{u})]s) = g^{j}_{\alpha\beta}([(us,\tilde{u})]) = g_\alpha(us)\tilde{g}_\beta(\tilde{u}) = g^{j}_{\alpha\beta}([(u,\tilde{u})])s.$$

Our goal is now to show that $\mathsf{P}(M, S^1)$ with the operation + is in fact an Abelian group. The proof is based on Blair [6].

Proposition 2.3.1. ($\mathsf{P}(M, S^1)$, +) *is an Abelian group.*

Proof. First we prove that the identity element of $\mathsf{P}(M, S^1)$ is given by the trivial bundle $P_0 = M \times S^1$. In fact, take $\pi : P \to M$ an element of $\mathsf{P}(M, S^1)$. Define the map α by

$$\begin{aligned} \alpha : \quad P &\longrightarrow P + P_0 \\ u &\longrightarrow [(u, (\pi(u), 1))]. \end{aligned}$$

We have that α is a bundle homomorphism because of the relation

$$\alpha(us) = [(us, (\pi(us), 1))] = [(us, (\pi(u), 1))] = [(u, (\pi(u), 1))]s = \alpha(u)s.$$

We claim that this α is injective. Indeed, if $\alpha(u) = \alpha(v)$, then $[(u,(\pi(u),1))] = [(v,(\pi(v),1))]$. Since $(u, (\pi(u), 1)) \sim (v, (\pi(v), 1))$, there exists $s \in S^1$ such that $v = us$ and $(\pi(v), 1) = (\pi(u), 1)s^{-1}$. From the second equality we have $s = 1$ and we get $u = v$.

To verify that α is onto, take an element $[(u, (m, s))] \in P + P_0$. As $(u, (m, s)) \sim (us, (m, 1))$, then we have $\alpha(us) = [(us, (m, 1))] = [(u, (m, s))]$, and α is surjective.

For the existence of an inverse, take $\pi : P \to M$ a element of $\mathsf{P}(M, S^1)$. We define the inverse element $-P$ in $\mathsf{P}(M, S^1)$ as follows. Consider the total space $-P$ as P, but with the action of S^1 given by $u \cdot s = us^{-1}$. It means that if we think of the fiber on P moving in a direction, then, in contrast, the fibers on $-P$ will be moving in the opposite.

We should verify that $P + (-P) \cong P_0$, where P_0 is the identity element defined above. For this, we define the map ϕ by

$$\begin{aligned} \phi\colon \quad P + (-P) &\longrightarrow P_0 \\ [(u, \tilde{u})] &\longrightarrow \phi([(u, \tilde{u})]) = (\pi(u), s) \end{aligned}$$

where $u = \tilde{u}s$.

First, let us see that ϕ is well defined. If $(u, \tilde{u}) \sim (v, \tilde{v})$, then there exists $t \in S^1$, such that $v = ut$ and $\tilde{v} = \tilde{u} \cdot t^{-1} = \tilde{u}t$. Since $\phi([(v, \tilde{v})]) = (\pi(v), r)$, where $v = \tilde{v}r$, and $\pi(u) = \pi(v)$, then we only need to show $r = s$, where $u = \tilde{u}s$. For this we have

$$v = ut = (\tilde{u}s)t = (\tilde{u}t)s = (\tilde{u} \cdot t^{-1})s = \tilde{v}s,$$

and so $r = s$.

Notice that ϕ is a bundle homomorphism. Indeed, if we take $[(u, \tilde{u})]$ in $P + (-P)$, with $u = \tilde{u}s$ and $t \in S^1$, we get

$$\phi([(u, \tilde{u})]t) = \phi([(ut, \tilde{u})]) = st = \phi([(u, v)])t.$$

We will prove that ϕ is bijective. The injectivity of ϕ is obtained as follows. If $\phi([(u, \tilde{u})]) = \phi([(v, \tilde{v})])$, then $(\pi(u), s) = (\pi(v), t)$, where $u = \tilde{u}s$ and $v = \tilde{v}t$. Because of $t = s$, we get $(u, \tilde{u}) \sim (v, \tilde{v})$.

For the surjectivity of ϕ, we take $(m, s) \in P_0$. Choosing any $u \in \pi^{-1}(m)$, we obtain $\phi([(us, u)]) = (m, s)$.

To show that the operation + is associative, we give three elements P_1, P_2 and P_3 in $P\,(M, S^1)$. We define the map ψ by

$$\begin{aligned} \psi: \quad (P_1 + P_2) + P_3 &\longrightarrow P_1 + (P_2 + P_3) \\ [([(u_1, u_2)], u_3)] &\longrightarrow [(u_1, [(u_2, u_3)])]. \end{aligned}$$

Clearly ψ is well defined. It is also easy to check that is injective and surjective. Furthermore, ψ is a bundle homomorphism. In fact, we have

$$\begin{aligned} \psi([([(u_1, u_2)], u_3)]t) &= \psi([([(u_1 t, u_2)], u_3)]) \\ &= [(u_1 t, [(u_2, u_3)])] \\ &= [(u_1, [(u_2, u_3)])]t \\ &= \psi([([(u_1, u_2)], u_3)])t. \end{aligned}$$

Finally, to prove the commutativity of + we exhibit directly the bundle homomorphism $\alpha : P + \tilde{P} \to \tilde{P} + P$, as $\alpha([(u, \tilde{u})]) = [(\tilde{u}, u)]$. We can easily to show that α is bijective. Therefore we have $P + \tilde{P} \cong \tilde{P} + P$. □

We write the sheaves $C^\infty(M, S^1)$, $C^\infty(M, \mathsf{R})$, and $C^\infty(M, \mathsf{Z})$ as S^1, R and Z, respectively. We show that the group $\mathsf{P}(M, S^1)$ is isomorphic to cohomology group $H^2(M, \mathsf{Z})$. For that, we prove first the following technical proposition. The proof is based on Blair [6] and Trang [28].

Proposition 2.3.2. *The group* $\mathsf{P}(M, S^1)$ *is isomorphic to* $H^1(M, \mathsf{S}^1)$.

Proof. Take $\pi : P \to M$ in $\mathsf{P}(M, S^1)$ and $\mathsf{U} = \{(U_i, \psi_i)\}$ a trivialization of P. For two elements (U_i, ψ_i) and (U_j, ψ_j) we have defined the transition maps g_{ij} in the section above by $g_{ij}(m) = g_{ij}(p)$, where $m \in U_i \cap U_j$, $p \in \pi^{-1}(m)$ and $g_{ij}(p) = g_i(p)g_j^{-1}(p)$. As we have already seen, these transition maps (g_{ij}) verify the cocycle condition. Thus, we have that (g_{ij}) is an element of the set of 1-chains $C^1(\mathsf{U}, \mathsf{S}^1)$. Let us see that $(g_{ij}) \in Z^1(\mathsf{U}, \mathsf{S}^1)$. For this we show $\delta_1(g_{ij}) = 1$: since $g_{ij} = g_i g_j^{-1}$, we have

$$g_{ijk} = \delta_1(g_{ij}) = g_{jk}\, g_{ik}^{-1} g_{ij} = g_j\, g_k^{-1}(g_i g_k^{-1})^{-1} g_i g_j^{-1} = 1.$$

As a consequence $[g_{ij}]$ defines an element of $H^1(M, \mathsf{S}^1)$. This assignment $P \mapsto [g_{ij}]$ is independent of the representative chosen in $\mathsf{P}(M, S^1)$. In fact, if $\tilde{P}$ is isomorphic to P, with trivializations $\mathsf{V} = \{(V_i, \varphi_i)\}$ and transition maps $h_{ij} : V_i \cap V_j \to S^1$, then we can construct a common refinement

$\mathsf{W} = \{W_i\}$ of $\{U_i\}$ and $\{V_i\}$. This allows us to define trivializations $(W_i, \overline{\psi}_i)$ and $(W_i, \overline{\varphi}_i)$ for P and $\widetilde{P}$, respectively, whose transition maps g_{ij} and h_{ij} satisfy $g_{ij}h_{ij}^{-1} \in B(\mathsf{W}, \mathsf{S}^1)$. Since we can define functions f_i with values in S^1 such that $h_i = f_i g_i$, we obtain

$$\begin{aligned} f_{ij} = \delta f_i = f_j f_i^{-1} = h_j g_j^{-1}(h_i g_i^{-1})^{-1} &= h_j g_j^{-1} g_i h_i^{-1} \\ &= h_j g_{ij} h_i^{-1} \\ &= g_{ij} h_{ij}^{-1}. \end{aligned}$$

Thus, we get $[g_{ij}] = [h_{ij}]$.

On the other hand, if we take an element $[g_{ij}] \in H^1(\mathsf{U}, \mathsf{S}^1)$, we can construct a principal S^1-bundle P with (g_{ij}) as transition maps. This principal S^1-bundle is given by

$$P = \frac{\bigsqcup_i U_i \times S^1}{\sim},$$

where the equivalence relation $\sim$ is defined as $(m, g) \sim (m, h)$ if and only if $h = g_{ij}(m)g$. Then, if we take another $(h_{ij}) \in [g_{ij}]$, this establishes also a principal S^1-bundle $\widetilde{P}$ on M. Let us see why $\widetilde{P}$ is isomorphic to P. Because of $(h_{ij}) \sim (g_{ij})$, for some $(f_i) \in C^0(\mathsf{U}, \mathsf{S}^1)$ we have $h_i = f_i g_i$. Then we define the isomorphism φ between P and $\widetilde{P}$ by $\varphi([(m, g)]) = [(m, f_i(m)g)]$. $\square$

On the other hand, a short exact sequence

$$0 \to \mathsf{Z} \xrightarrow{i} \mathsf{R} \xrightarrow{exp} S^1 \to 0$$

induces a short exact sequence of sheaves

$$0 \longrightarrow \mathsf{Z} \longrightarrow \mathsf{R} \longrightarrow \mathsf{S}^1 \longrightarrow 0.$$

For an open covering $\mathsf{U} = \{U_i\}$ of M, the above sequence induces the exact cohomology sequence

$$\cdots \to H^1(\mathsf{U}, \mathsf{R}) \to H^1(\mathsf{U}, \mathsf{S}^1) \to H^2(\mathsf{U}, \mathsf{Z}) \to H^2(\mathsf{U}, \mathsf{R}) \to \cdots.$$

Therefore to conclude that $\mathsf{P}(M, S^1)$ is isomorphic to $H^2(M, \mathsf{Z})$ it is enough to prove $H^1(\mathsf{U}, \mathsf{R}) \cong H^2(\mathsf{U}, \mathsf{R}) \cong 0$.

Proposition 2.3.3. *We have* $H^2(\mathsf{U}, \mathsf{R}) \cong 0$.

Proof. We take $\mathcal{U} = \{U_i\}$ as a locally finite open cover of M. Since $\mathbb{R}$ is a fine sheaf (see Warner [30]), we have a partition of unity for $\mathbb{R}$ subordinate to the open cover $\{U_i\}$, which is denoted by $\{\varphi_i\}$. Writing $V_i = supp(\varphi_i)$, we have $V_i \subset U_i$ and $\sum_i \varphi_i = 1$. To show that $H^2(\mathcal{U}, \mathbb{R}) \cong 0$ we must prove the equality $B^2(\mathcal{U}, \mathbb{R}) = Z^2(\mathcal{U}, \mathbb{R})$. It is clear that $B^2(\mathcal{U}, \mathbb{R}) \subset Z^2(\mathcal{U}, \mathbb{R})$. On the other hand, taking $(f_{ijk}) \in Z^2(\mathcal{U}, \mathbb{R})$, we must find $(g_{ij}) \in C^1(\mathcal{U}, \mathbb{R})$ such that $\delta(g_{ij}) = f_{ijk}$. For that, we define $g_{ij} = \sum_k \varphi_k f_{ijk}$. Then we compute

$$\begin{aligned}\delta(g_{ij}) &= g_{jk} - g_{ik} + g_{ij} \\ &= \sum_l \varphi_l f_{jkl} - \sum_l \varphi_l f_{ikl} + \sum_l \varphi_l f_{ijl} \\ &= \sum_l \varphi_l (f_{jkl} - f_{ikl} + f_{ijl}).\end{aligned}$$

Since $(f_{ijk}) \in Z^2(\mathcal{U}, \mathbb{R})$, we get $\delta(f_{ijk}) = f_{jkl} - f_{ikl} + f_{ijl} - f_{ijk} = 0$. Therefore $f_{ijk} = f_{jkl} - f_{ikl} + f_{ijl}$. Replacing this in $\delta(g_{ij})$, we arrive to

$$\delta(g_{ij}) = \sum_l \varphi_l f_{ijk} = \left(\sum_l \varphi_l\right) f_{ijk} = f_{ijk}.$$

Thus $(f_{ijk}) \in B^2(\mathcal{U}, \mathbb{R})$ and as a consequence $Z^2(\mathcal{U}, \mathbb{R}) = B^2(\mathcal{U}, \mathbb{R})$. □

Proposition 2.3.4. *The group* $H^1(\mathcal{U}, \mathbb{R})$ *is isomorphic to* 0.

Proof. We should prove $Z^1(\mathcal{U}, \mathbb{R}) = B^1(\mathcal{U}, \mathbb{R})$. In a similar way as the above proof, for $f_{ij} \in Z^1(\mathcal{U}, \mathbb{R})$ we define $g_i = \sum_l \varphi_l f_{il}$. The proof follows easily. □

By Proposition 1.2.11, we have that the sequence of cohomology

$$\cdots \to H^1(M, \mathbb{R}) \to H^1(M, \mathbb{S}^1) \to H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R}) \to \cdots$$

is exact. As $H^2(\mathcal{U}, \mathbb{R}) \cong H^1(\mathcal{U}, \mathbb{R}) \cong 0$, we obtain by the Leray theorem that $H^2(M, \mathbb{R}) \cong H^1(M, \mathbb{R}) \cong 0$. Since the sequence in cohomology is exact, we arrive to $H^1(M, \mathbb{S}^1) \cong H^2(M, \mathbb{Z})$. Therefore, by Proposition 2.3.2, we conclude $\mathcal{P}(M, S^1) \cong H^2(M, \mathbb{Z})$. Finally, since $\mathbb{Z}$ is a discrete group, we have $H^2(M, \mathbb{Z}) \cong H^2(M, \mathbb{Z})$. As a consequence we obtain $\mathcal{P}(M, S^1) \cong H^2(M, \mathbb{Z})$.

Chapter 3

Riemannian Geometry on Principal S^1-bundles

This chapter focuses on the study of the Riemannian structure on principal S^1-bundles and afterwards in the construction of these on a connected complete Riemannian manifold M. In fact, S. Kobayashi himself established the following result.

"Let M be a complete Kähler manifold with Kählerian pinching $> \delta$. Then there exists a principal circle bundle P over M and a Riemannian metric on P with Riemannian pinching $> \dfrac{\delta}{(4-3\delta)}$."

As we will see, the relation that we will find between the sectional curvatures of the total space P and that of the base space M allows us to obtain the above result.

In the first section of this chapter we will construct a metric on P induced by a metric on M. Next we will give the relation between the connection forms and the curvature forms of P and M with respect to these metrics. The second section is dedicated to find the relation between the sectional curvatures of P and M. Finally, in the last section we build the total space P with the features given in the result stated above.

3.1 Riemannian connections and curvature on principal S^1-bundles

Let $\pi : P \to M$ be a principal S^1-bundle, and γ a connection form on P. Recall that we can always define a normal coordinate system around x in M; that is, an orthonormal frame $\{E^1, ..., E^n\}$ on a neighbourhood U of x. Consider $\{\theta^1, ..., \theta^n\}$ its dual basis. Then we express the Riemannian metric on M as

$$ds^2 = \sum_{i=1} (\theta^i)^2.$$

The objective of this section is to study the curvature on P with respect to the Riemannian metric

$$d\sigma^2 = \pi^*(ds^2) + (ab\gamma)^2,$$

where a and b are real numbers.

Remark 3.1.1. In the above expression, the constants a and b will play an important role in the construction of principal S^1-bundle required. The constant b is necessary to rescale the connection form γ in order that this will end up being the pullback, through π, of an integral form on M (which is necessary in the proof of theorem of Kobayashi, as we explain throughout this chapter). On the other hand, the constant a is determined to define an Einstein metric $d\sigma^2$ on P. The latter will be explained in detail in the next chapter.

Notice that the non-negative bilinear form $d\sigma^2$ defines a metric on P. It is enough to verify that $d\sigma^2(u, u) = 0$ if and only if $u = 0$ in T_pP. In fact, since $d\sigma^2(u, u) = ds^2(\pi_*(u), \pi_*(u)) + (ab)^2\gamma^2(u)$, we have that $d\sigma^2(u, u) = 0$ if and only if $ds^2(\pi_*(u), \pi_*(u)) = 0$ and $\gamma(u) = 0$. As $ds^2(\pi_*(u), \pi_*(u)) = 0$ is equivalent to $u \in \ker(\pi_*) = V_p$, and $\gamma(u) = 0$ holds when $u \in H_p$, we conclude that $u = 0$.

Next we study the Riemannian connection induced by the metric $d\sigma^2$ on P. Let ∇ be the Riemannian connection for the metric ds^2 on M and (ω_j^i) the skew-symmetric matrix of 1-forms on $U \subset M$ defined by ∇. The 1-forms $\omega^i{}_j$ are subject to

$$\nabla_X E^j = \sum_i \omega_j^i(X)E^i,$$

or equivalently

$$\omega_j^i(X) = \theta^i(\nabla_X E^j).$$

We call (ω_j^i) the *connection form* of the metric ds^2.

Proposition 3.1.2 (First structure equation)**.** *On the neighbourhood $U \subset M$, the equality*

$$d\theta^i = -\sum_j \omega_j^i \wedge \theta^j$$

holds.

Proof. Taking $X = \sum_j X_j E^j$ and $Y = \sum_j Y_j E^j$ as vector fields on M, we have

$$\begin{aligned}
d\theta^i(X, Y) &= \frac{1}{2}\Big\{X(\theta^i(Y)) - Y(\theta^i(X)) - \theta^i([X, Y])\Big\} \\
&= \frac{1}{2}\Big\{X(Y_i) - Y(X_i) - \theta^i(\nabla_X Y - \nabla_Y X)\Big\} \\
&= \frac{1}{2}\{X(Y_i) - Y(X_i) - \theta^i(\sum_j (X(Y_j)E^j + Y_j \nabla_X E^j) \\
&\qquad - \sum_j (Y(X_j)E^j + X_j \nabla_Y E^j))\} \\
&= \frac{1}{2}\{X(Y_i) - Y(X_i) - X(Y_i) - \theta^i(\sum_j Y_j \nabla_X E^j) + Y(X_i) \\
&\qquad + \theta^i(\sum_j X_j \nabla_Y E^j)\} \\
&= \frac{1}{2}\theta^i(\sum_j (X_j \nabla_Y E^j - Y_j \nabla_X E^j)).
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
\omega_j^i \wedge \theta^j(X, Y) &= \frac{1}{2}\{\omega_j^i(X)\theta^j(Y) - \omega_j^i(Y)\theta^j(X)\} \\
&= \frac{1}{2}\{\theta^i(\nabla_X E^j)Y^j - \theta^i(\nabla_Y E^j)X^j\} \\
&= \frac{1}{2}\theta^i(Y^j \nabla_X E^j - X^j \nabla_Y E^j).
\end{aligned}$$

Comparing both quantities , we get

$$d\theta^i = -\sum_j \omega_j^i \wedge \theta^j.$$

□

Consider the curvature R of ∇. We write its components with respect to $\theta^1, \dots, \theta^n$ as K_{ijkl}, or, in other words, we set

$$R(E^k, E^l)E^j = \sum_i K_{ijkl}E^i.$$

Then, we have

$$K_{ijkl} = ds^2(R(E^k, E^l)E^j, E^i).$$

We define the *Riemannian curvature tensor* as

$$R^\nabla(X, Y, Z, W) = ds^2(R(Z, W)Y, X).$$

Lemma 3.1.3. *The Riemannian curvature tensor* $R^\nabla(X, Y, Z, W)$ *satisfies the following properties:*

(i) $R^\nabla(X, Y, Z, W) = -R^\nabla(Y, X, Z, W)$,

(ii) $R^\nabla(X, Y, Z, W) = -R^\nabla(Y, X, W, Z)$,

(iii) $R^\nabla(X, Y, Z, W) = R^\nabla(Z, W, X, Y)$.

Proof. See Lee [20] □

Lemma 3.1.4 (Bianchi's identities)**.** *Let* R *be the curvature of* ∇. *Then we have*

(i) [Bianchi's first identity]

$$R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0;$$

(ii) [Bianchi's second identity]

$$(\nabla_Z R)(X, Y)W + (\nabla_X R)(Y, Z)W + (\nabla_Y R)(Z, X)W = 0.$$

Proof. See Lee [20] □

Using the above lemmas, we conclude that K_{ijkl} satisfies

$$K_{ijkl} = -K_{jikl} = -K_{ijlk} = K_{klij},$$

and

$$K_{ijkl} + K_{iklj} + K_{iljk} = 0.$$

We define the *curvature 2-form* (Ω^i_j) with respect to the Riemannian connection ∇ as

$$\Omega^i_j = \frac{1}{2}\sum_{k,l} K_{ijkl}\,\theta^k \wedge \theta^l.$$

Let us see how the curvature 2-form (Ω^i_j) and the connection (ω^i_j) are related.

Proposition 3.1.5 (Second structure equation.)**.** *Let ∇ be the connection induced by ds^2 on M, and (Ω^i_j), (ω^i_j) its curvature and connection form, respectively. Then we have*

$$d\omega^i_j = -\sum_k \omega^i_k \wedge \omega^k_j + \Omega^i_j.$$

Proof. For $X = \sum_l X_l E^l$ and $Y = \sum_l Y_l E^l$ vector fields on U, we get

$$\begin{aligned}
\Omega^i_j(X,Y) &= \frac{1}{2}\sum_{k,l} K_{ijkl}\,\theta^k \wedge \theta^l(X,Y)\\
&= \frac{1}{2}\sum_{k,l} K_{ijkl}\,\frac{1}{2}(X_k Y_l - X_l Y_k)\\
&= \frac{1}{4}\sum_{k,l} K_{ijkl} X_k Y_l - \frac{1}{4}\sum_{k,l} K_{ijkl} X_l Y_k\\
&= \frac{1}{4}\sum_{k,l} K_{ijkl} X_k Y_l + \frac{1}{4}\sum_{k,l} K_{ijkl} X_k Y_l\\
&= \frac{1}{2}\sum_{k,l} K_{ijkl} X_k Y_l\\
&= \frac{1}{2} ds^2(R(X,Y)E^j, E^i).
\end{aligned}$$

Now, calculating $R(X,Y)E^j$ yields

$$\begin{aligned}
R(X,Y)E^i &= \nabla_X\nabla_Y E^j - \nabla_Y\nabla_X E^j - \nabla_{[X,Y]}E^j\\
&= \nabla_X\Big(\sum_k \omega^k_j(Y)E^k\Big) - \nabla_Y\Big(\sum_k \omega^k_j(X)E^k\Big) - \sum_k \omega^k_j([X,Y])E^k\\
&= \sum_k \Big(X(\omega^k_j(Y))E^k + \omega^k_j(Y)\nabla_X E^k\Big)\\
&\quad - \sum_k \Big(Y(\omega^k_j(X))E^k + \omega^k_j(X)\nabla_Y E^k\Big) - \sum_k \omega^k_j([X,Y])E^k\\
&= \sum_k \big(X(\omega^k_j(Y)) - Y(\omega^k_j(X)) - \omega^k_j([X,Y])\big)E^k +\\
&\quad \sum_k \big(\omega^k_j(Y)\nabla_X E^k - \omega^k_j(X)\nabla_Y E^k\big)
\end{aligned}$$

Then we have

$$
\begin{aligned}
\Omega^i_j(X,Y) &= \frac{1}{2}ds^2(R(X,Y)E^j, E^i)) \\
&= \frac{1}{2}ds^2(\sum_k (X(\omega^k_j(Y)) - Y(\omega^k_j(X)) - \omega^k_j([X,Y]))E^k, E^i) + \\
&\quad \frac{1}{2}ds^2(\sum_k (\omega^k_j(Y)\nabla^X E^k - \omega^k_j(X)\nabla^Y E^k), E^i) \\
&= \frac{1}{2}\{X(\omega^i_j(Y)) - Y(\omega^i_j(X)) - \omega^i_j([X,Y])\} + \\
&\quad \frac{1}{2}ds^2(\sum_k \omega^k_j(Y)\sum_l(\omega^l_k(X)E^l) - \sum_k \omega^k_j(X)\sum_l(\omega^l_k(Y)E^l), E^i) \\
&= d\omega^i_j(X,Y) + \frac{1}{2}ds^2(\sum_{k,l}(\omega^k_j(Y)\omega^l_k(X) - \omega^k_j(X)\omega^l_k(Y))E^l, E^i) \\
&= d\omega^i_j(X,Y) + \frac{1}{2}\sum_k(\omega^k_j(Y)\omega^i_k(X) - \omega^k_j(X)\omega^i_k(Y)) \\
&= d\omega^i_j(X,Y) + \sum_k \omega^i_k \wedge \omega^k_j(X,Y).
\end{aligned}
$$

Therefore we get

$$\Omega^i_j = d\omega^i_j + \sum_k \omega^i_k \wedge \omega^k_j.$$

□

On the other hand, using Proposition 2.1.14, we have the following equality for the connection form γ

$$\Gamma = d\gamma + \frac{1}{2}[\gamma, \gamma],$$

where Γ represents the curvature form of γ. Since S^1 is a commutative group, in the above formula we obtain $\Gamma = d\gamma$. In addition, by Proposition 2.1.16, we can express Γ as the pullback through π of a 2-form on U. More specifically we can write

$$\Gamma = \pi^*(\sum_{i,j=1}^{2} A_{ij}\theta^i \wedge \theta^j),$$

where $A_{ij} = -A_{ji}$.

Next we find the Riemannian connection for the metric $d\sigma^2$ defined on P. For sake of simplicity, we write the components $ab\gamma$ and $\pi^*\theta^i$ of $d\sigma^2$ as

$$\phi^0 = ab\gamma,$$

$$\phi^i = \pi^*\theta^i.$$

With this convention we have

$$d\sigma^2 = \sum_{a=0} (\phi^a)^2.$$

We denote by (ψ^a_β) the connection form of the metric $d\sigma^2$ on P. Before we construct the connection form of the metric $d\sigma^2$ on P, we establish several results that will allow us determine explicitly the mentioned connection form.

Lemma 3.1.6. *Let (M^n, ds^2) be a Riemannian manifold and $\{U_\lambda\}_{\lambda\in\Lambda}$ an open covering of M. Consider (ξ^j) and $(\tilde{\xi}^j)$ orthonormal local frames on U_λ and U_τ, with $\lambda, \tau \in \Lambda$, respectively, and*

$$\tilde{\xi}^i = \sum_j s^i_j \xi^j \quad \text{on } U_\lambda \cap U_\tau.$$

If there exists 1-forms (ω^i_j) and $(\tilde{\omega}^i_j)$ on U_λ and U_τ, such that

$$\tilde{\omega}^i_j = \sum_{k,l} s^i_k \omega^k_l s^j_l - \sum_k ds^i_k \cdot s^j_k \tag{3.1}$$

on $U_\lambda \cap U_\tau$, then these 1-forms $(\tilde{\omega}^i_j)$ and (ω^i_j) determine a linear connection on M.

Proof. Let $\pi : L(M) \to M$ be the linear frame bundle defined in Section 2.2, and $\omega_\lambda = (\omega^i_j)$, $\omega_\tau = (\tilde{\omega}^i_j)$ the $\mathfrak{gl}_n(\mathbb{R})$-valued forms on U_λ and U_τ, respectively, that satisfy Equation (3.1). By hypothesis, as $s = (s^i{}_j)$ is an orthonormal matrix, we have

$$\omega_\tau = s\omega_\lambda s^{-1} - ds \cdot s^{-1}.$$

On the other hand, for any member $U_{\tilde{\lambda}}, U_{\tilde{\tau}}$ of the covering $\{U_\lambda\}_{\lambda\in\Lambda}$, we have on $U_{\tilde{\lambda}} \cap U_{\tilde{\tau}}$ a map $s_{\tilde{\lambda}\tilde{\tau}} : U_{\tilde{\lambda}} \cap U_{\tilde{\tau}} \to GL_n(\mathbb{R})$ defined by $s_{\tilde{\lambda}\tilde{\tau}}(m) = (s^i_j(m))$, where (s^i_j) is given above. Since the collection of these maps $s_{\tilde{\lambda}\tilde{\tau}}$ satisfy the cocycle conditions, they induce local trivializations on the linear frame bundle $\pi : L(M) \to M$. As a consequence these local trivializations induce a principal $GL_n\mathbb{R}$-structure on this linear frame bundle.

Considering these local trivializations, the orthonormal local frames $\xi_\lambda = (\xi^j)$ and $\xi_\tau = (\tilde{\xi}^j)$ turn into canonical sections on U_λ and U_τ, respectively. Therefore, due to Remark 2.1.12, we obtain a connection form $\varphi = (\varphi^i{}_j)$ on $L(M)$, where locally it verifies $\omega_\lambda = \xi^*_\lambda\varphi$ on U_λ. By Proposition 2.2.1, this connection form φ induces a covariant derivative ∇ on M. Working as in the proof of Proposition 2.2.1, if we choose the same local sections ξ_λ on U_λ, the connection form of the covariant derivative ∇ on U_α consists of the same $\mathfrak{gl}_n(\mathbb{R})$-valued forms $\omega_\alpha = (\omega^i_j)$. $\square$

The converse also is true. This is proved next.

Proposition 3.1.7. *Let (ω^i_j) and $(\tilde{\omega}^i_j)$ be the local expression of a connection form on U_α and U_β, respectively. If the orthonormal local frames (ξ^j) and $(\tilde{\xi}^j)$ are related by*

$$\tilde{\xi}^i = \sum_j s^i_j \xi^j \quad \text{on } U_\alpha \cap U_\beta,$$

then we obtain

$$\tilde{\omega}^i_j = \sum_{k,l} s^i_k \omega^k_l s^j_l - \sum_k ds^i_k \cdot s^j_k \tag{3.2}$$

Proof. In fact, we consider the dual basis (θ^j) and $(\tilde{\theta}^j)$ of (ξ^j) and $(\tilde{\xi}^j)$, respectively; which are related by the equality

$$\tilde{\theta}^i = \sum_j s^i_j \theta^j. \tag{3.3}$$

Evaluating on a vector field X, we obtain

$$\begin{aligned}
\tilde{\omega}^i_j(X) = \tilde{\theta}^i(\nabla_X \tilde{\xi}^j) &= \sum_k s^i_k \theta^k(\nabla_X \tilde{\xi}^j) \\
&= \sum_k s^i_k \theta^k\Big(\nabla_X\Big(\sum_l s^j_l \xi^l\Big)\Big) \\
&= \sum_k s^i_k \theta^k\Big(\sum_l X(s^j_l)\xi^l + s^j_l \nabla_X \xi^l\Big) \\
&= \sum_k s^i_k \sum_l X(s^j_l)\theta^k(\xi^l) + \sum_k s^i_k \sum_l s^j_l \theta^k(\nabla_X \xi^l) \\
&= \sum_k s^i_k X(s^j_k) + \sum_k s^i_k \sum_l s^j_l \omega^k_l(X) \\
&= \sum_k s^i_k ds^j_k(X) + \sum_{k,l} s^i_k \omega^k_l(X) s^j_l
\end{aligned}$$

So we get

$$\tilde{\omega}^i_j = \sum_k s^i_k ds^j_k + \sum_{k,l} s^i_k \omega^k_l s^j_l.$$

Writing $s = (s^i_j)$, we have

$$s \cdot ds^t = -ds \cdot s^t = -ds \cdot s^{-1}.$$

If we apply this in the equation above, the required result is obtained immediately. $\square$

With respect to the curvature form Γ of γ, we observe that this can be written on U_a as

$$\Gamma = \pi^*\left(\sum_{i,j=1}^{n} A_{ij}\theta^i \wedge \theta^j\right).$$

Analogously, this is expressed on U_β as

$$\Gamma = \pi^*\left(\sum_{i,j=1}^{n} \tilde{A}_{ij}\tilde{\theta}^i \wedge \tilde{\theta}^j\right).$$

So, everything together derives in

$$\begin{aligned}
\sum_{i,j=1}^{n} A_{ij}\theta^i \wedge \theta^j &= \sum_{i,j=1}^{n} A_{ij}\Big(\sum_{k=1}^{n} s_i^k\tilde{\theta}^k\Big) \wedge \Big(\sum_{l=1}^{n} s_j^l\tilde{\theta}^l\Big) \\
&= \sum_{i,j=1}^{n} A_{ij} \sum_{k,l=1}^{n} s_i^k s_j^l \tilde{\theta}^k \wedge \tilde{\theta}^l \\
&= \sum_{k,l=1}^{n}\Big(\sum_{i,j=1}^{n} A_{ij}s_i^k s_j^l\Big)\tilde{\theta}^k \wedge \tilde{\theta}^l
\end{aligned}$$

Thus, the maps A_{ij} and $\tilde{A}_{ij}$ are related by

$$\tilde{A}_{kl} = \sum_{i,j=1}^{n} A_{ij}s_i^k s_j^l. \tag{3.4}$$

Returning to the metric $d\sigma^2$, we search for its connection form by means of the first structure equation. Since (ψ^α_β) is skew-symmetric, we have $\psi^\alpha_\alpha = 0$ for all $\alpha = 0, 1, ..., n$. Using the first structure equation, we have then

$$d\phi^0 = -\sum_{j=1}^{n} \psi^0_j \wedge \phi^j$$

Furthermore, since $\phi^0 = ab\gamma$, we get

$$\begin{aligned}
d\phi^0 = abd\gamma = ab\Gamma &= ab\pi^*\Big(\sum_{i,j=1}^{n} A_{ij}\theta^i \wedge \theta^j\Big) \\
&= ab\sum_{i,j=1}^{n} A_{ij}\pi^*\theta^i \wedge \pi^*\theta^j \\
&= ab\sum_{i,j=1}^{n} A_{ij}\phi^i \wedge \phi^j \\
&= \sum_{j=1}^{n}\Big(\sum_{i=1}^{n} abA_{ij}\phi^i\Big) \wedge \phi^j.
\end{aligned}$$

Then we have

$$\psi^0_j = -\sum_{i=1}^{n} abA_{ij}\phi^i = \sum_{i=1}^{n} abA_{ji}\phi^i. \tag{3.5}$$

In a similar way we find ψ^i_j, when $i, j \in \{1, ..., n\}$. Because of $\phi^i = \pi^*\theta^i$, we obtain

$$\begin{aligned} d\phi^i = d(\pi^*\theta^i) = \pi^*(d\theta^i) &= \pi^*(-\sum_{j=1}^{n} \omega^i_j \wedge \theta^j) \\ &= -\sum_{j=1}^{n} \pi^*\omega^i_j \wedge \pi^*\theta^j \\ &= -\sum_{j=1}^{n} \pi^*\omega^i_j \wedge \phi^j \end{aligned}$$

On the other hand, using the first structure equation, we can also express $d\phi^i$ as

$$d\phi^i = -\sum_{\alpha=0}^{n} \psi^i_\alpha \wedge \phi^\alpha = -\psi^i_0 \wedge \phi^0 - \sum_{j=1}^{n} \psi^i_j \wedge \phi^j.$$

Using Equation 3.5, we get

$$\begin{aligned} d\phi^i &= (\sum_{j=1}^{n} abA_{ij}\phi^j) \wedge ab\gamma - \sum_{j=1}^{n} \psi^i_j \wedge \phi^j \\ &= -\sum_{j=1}^{n} (ab)^2 A_{ij}\gamma \wedge \phi^j - \sum_{j=1}^{n} \psi^i_j \wedge \phi^j \\ &= -\sum_{j=1}^{n} ((ab)^2 A_{ij}\gamma + \psi^i_j) \wedge \phi^j. \end{aligned}$$

Comparing the previous results we arrive to

$$\pi^*\omega^i_j = (ab)^2 A_{ij}\gamma + \psi^i_j = abA_{ij}\phi^0 + \psi^i_j.$$

Therefore, we have

$$\psi^i_j = \pi^*\omega^i_j - abA_{ij}\phi^0. \tag{3.6}$$

Henceforth, the indices α, β with run over $\{0, 1, 2, ..., n\}$.

Theorem 3.1.8. *The Riemannian connection on P with respect to $d\sigma^2$ is given by (ψ^α_β) (as described in Equations (3.5) and (3.6)). In brief, we have*

$$\psi^0_0 = 0,$$

$$\psi^i_0 = -\psi^0_i = -\sum_{j=1}^{n} abA_{ij}\phi^j,$$

$$\psi^i_j = \pi^*(\omega^i_j) - abA_{ij}\phi^0.$$

Proof. Because of the isomorphism $H_p \cong T_{\pi(p)}M$, each element E^i from an orthonormal frame of M induces a vector field ξ^i on P. In addition, considering v_0 as the generator of vertical subspace $V_p \subset T_pP$, we define the vector field ξ^0 as $\frac{1}{ab}v_0$. We obtain a orthonormal local frame $\{\xi^0, \xi^1, ..., \xi^n\}$ on the neighborhood $V = \pi^{-1}(U) \subset P$.

Let us see that (ψ^α_β) defines a linear connection on P. For this we use Lemma 3.1.6.

It is clear that $\{\phi^\alpha\}$ is the dual basis of $\{\xi^\alpha\}$. Consider the orthonormal local frame $\{\tilde{\xi}^\alpha\}$ and its dual basis $\{\tilde{\phi}^\alpha\}$ on another neighbourhood $\tilde{V} = \pi^{-1}(\tilde{U}) \subset P$, which are defined in a similar way as above. We also define $(\tilde{\psi}^\alpha_\beta)$ on $\tilde{V}$ in the same way as (ψ^α_β).

To show that (ψ^α_β) is a linear connection on P, it is sufficient to establish the following equality

$$\tilde{\psi}^\alpha_\beta = \sum_{\lambda,\mu=0}^{n} t^\alpha_\lambda \psi^\lambda_\mu t^\beta_\mu - \sum_{\lambda=0}^{n} dt^\alpha_\lambda \cdot t^\beta_\lambda \quad \text{on } V \cap \tilde{V}, \tag{3.7}$$

where the (t^α_β) are defined so that

$$\tilde{\xi}^\alpha = \sum_\beta t^\alpha_\beta \xi^\beta \quad \text{holds on } V \cap \tilde{V}.$$

By Linear Algebra, we notice that (t^α_β) also relates the dual basis $\{\phi^\alpha\}$ and $\{\tilde{\phi}^\alpha\}$ as follows

$$\tilde{\phi}^\alpha = \sum_\beta t^\alpha_\beta \phi^\beta. \tag{3.8}$$

Clearly, we have $t^0_0 = 1$ and $t^i_0 = t^0_i = 0$. For the other cases, we get

$$\tilde{\phi}^i = \pi^*\tilde{\theta}^i = \pi^*\left(\sum_j s^i_j\theta^j\right) = \sum_j (s^i_j \circ \pi)\pi^*\theta^j = \sum_j (s^i_j \circ \pi)\phi^j.$$

Comparing with (3.8), we arrive to $t^i_j = s^i_j \circ \pi$.

Now we prove Equation (3.7). For ψ_0^0, it is clear. Let us check the expression for ψ_0^i. In this case, Equation (3.5) reduces to

$$\tilde{\psi}_0^i = \sum_{j=1}^{n} t_j^i \psi_0^j.$$

From here we get

$$\begin{aligned}
\sum_{j=1}^{n} t_j^i \psi_0^j &= \sum_{j=1}^{n} t_j^i \Big(-\sum_{k=1}^{n} abA_{jk}\phi^k\Big) \\
&= -ab \sum_{j,k=1}^{n} t_j^i A_{jk}\phi^k \\
&= -ab \sum_{j,k=1}^{n} t_j^i A_{jk} \sum_{l=1}^{n} t_k^l \tilde{\phi}^l \\
&= -ab \sum_{l=1}^{n} \Big(\sum_{j,k=1}^{n} t_j^i A_{jk} t_k^l\Big)\tilde{\phi}^l.
\end{aligned}$$

Using Equation (3.4) we obtain

$$\sum_{j=1}^{n} t_j^i \psi_0^j = -ab \sum_{l=1}^{n} \tilde{A}_{il}\tilde{\phi}^l = \tilde{\psi}_0^i,$$

as desired.

Finally, for ψ_j^i we have

$$\begin{aligned}
\tilde{\psi}_j^i &= \pi^*(\tilde{\omega}_j^i) - ab\tilde{A}_{ij}\tilde{\phi}^0 \\
&= \pi^*\Big(\sum_{k,l=1}^{n} s_k^i \omega_l^k t_j^l - \sum_{k=1}^{n} ds_k^i \cdot s_j^k\Big) - ab\tilde{A}_{ij}\tilde{\phi}^0 \\
&= \sum_{k,l=1}^{n} t_k^i \pi^*\omega_l^k t_j^l - \sum_{k=1}^{n} \pi^*(ds_k^i \cdot s_j^k) - ab\tilde{A}_{ij}\tilde{\phi}^0.
\end{aligned}$$

Using Equation (3.6) for $\pi^*\omega^k{}_l$ we obtain

$$\begin{aligned}
\tilde{\psi}_j^i &= \sum_{k,l=1}^{n} t_k^i(\psi_l^k + abA_{kl}\phi^0)t_j^l - \sum_{k=1}^{n} \pi^*(ds_k^i \cdot s_j^k) - ab\tilde{A}_{ij}\tilde{\phi}^0 \\
&= \sum_{k,l=1}^{n} t_k^i \psi_l^k t_j^l + ab \sum_{k,l=1}^{n} t_k^i A_{kl} t_j^l \phi^0 - \sum_{k=1}^{n} \pi^*(ds_k^i \cdot s_j^k) - ab\tilde{A}_{ij}\phi^0.
\end{aligned}$$

By Equation (3.4), if we replace $\sum_{k,l=1}^{n} t^i_k A_{kl} t^j_l$ by $\tilde{A}_{ij}$, we get

$$
\begin{aligned}
\tilde{\psi}^i_j &= \sum_{k,l=1}^{n} t^i_k \psi^k_l t^j_l + ab\tilde{A}_{ij}\phi^0 - \sum_{k=1}^{n} \pi^*(ds^i_k \cdot s^j_k) - ab\tilde{A}_{ij}\phi^0 \\
&= \sum_{k,l=1}^{n} t^i_k \psi^k_l t^j_l - \sum_{k=1}^{n} \pi^*(ds^i_k \cdot s^j_k) \\
&= \sum_{k,l=1}^{n} t^i_k \psi^k_l t^j_l - \sum_{k=1}^{n} \pi^*(ds^i_k) \cdot \pi^*(s^j_k) \\
&= \sum_{k,l=1}^{n} t^i_k \psi^k_l t^j_l - \sum_{k=1}^{n} d\pi^*(s^i_k) \cdot t^j_k \\
&= \sum_{k,l=1}^{n} t^i_k \psi^k_l t^j_l - \sum_{k=1}^{n} dt^i_k \cdot t^j_k.
\end{aligned}
$$

Therefore, by Lemma 3.1.6, (ψ^α_β) defines a linear connection on P . Moreover, the covariant derivative induced by (ψ^α_β) has as connection form this same matrix (ψ^α_β).

We still need to show that this connection is Riemannian. For this, we only require that the covariant derivative to be torsion free. But this condition is equivalent to the first structure equation, that (ψ^α_β) satisfies, so (ψ^α_β) indeed establishes a Riemannian connection. □

The curvature forms of M and P are related as follows.

Theorem 3.1.9. *If (Ψ^α_β) represents the curvature form of the connection (ψ^α_β) on P , then*

$$
\begin{aligned}
&\Psi^0_0 = 0, \\
&\Psi^i_0 = -\Psi^0_i = -a^2b^2 \sum_{k,l} A_{ik}A_{kl}\phi^l \wedge \phi^0 - ab \sum_{k,l} A_{il;k}\phi^k \wedge \phi^l, \text{ and} \\
&\Psi^i_j = \pi^*(\Omega^i_j) - \sum_{k,l} a^2b^2(A_{ij}A_{kl} + A_{ik}A_{jl})\phi^k \wedge \phi^l - ab \sum_{k} A_{ij;k}\phi^k \wedge \phi^0;
\end{aligned}
$$

where $\sum_k A_{ij;k}\theta^k = dA_{ij} - \sum_k A_{ik}\omega^k_j + \sum_k A_{kj}\omega^i_k$.

Proof. First, we prove this result for Ψ^i_0. Using Theorem 3.1.8 and the second structure equation, we have

$$
\begin{aligned}
\Psi^i_0 &= d\psi^i_0 + \sum_k \psi^i_k \wedge \psi^k_0 \\
&= -\sum_k abd(A_{ik}\phi^k) + \sum_k \{(\pi^*(\omega^i_k) - abA_{ik}\phi^0) \wedge (-\sum_l abA_{kl}\phi^l)\} \\
&= -\sum_k abdA_{ik} \wedge \phi^k - \sum_k abA_{ik}d\phi^k - \sum_{k,l} abA_{kl}\pi^*(\omega^i_k) \wedge \phi^l + \\
&\qquad \sum_{k,l} (ab)^2 A_{ik}A_{kl}\phi^0 \wedge \phi^l \\
&= -ab\left(\sum_k dA_{ik} \wedge \phi^k + \sum_k A_{ik}d\phi^k + \sum_{k,l} A_{kl}\pi^*(\omega^i_k) \wedge \phi^l\right) + \\
&\qquad \sum_{k,l} (ab)^2 A_{ik}A_{kl}\phi^0 \wedge \phi^l.
\end{aligned}
$$

Next we expand

$$
\begin{aligned}
\sum_{k,l} A_{il;k}\phi^k \wedge \phi^l &= \sum_l \Big(\sum_k A_{il;k}\phi^k\Big) \wedge \phi^l \\
&= \sum_l \pi^*\Big(\sum_k A_{il;k}\theta^k\Big) \wedge \phi^l \\
&= \sum_l \pi^*\Big(dA_{il} - \sum_k A_{ik}\omega^k_l + \sum_k A_{kl}\omega^i_k\Big) \wedge \phi^l \\
&= \sum_l \Big(dA_{il} - \sum_k A_{ik}\pi^*\omega^k_l + \sum_k A_{kl}\pi^*\omega^i_k\Big) \wedge \phi^l \\
&= \sum_l dA_{il} \wedge \phi^l - \sum_l \Big(\sum_k A_{ik}\pi^*\omega^k_l\Big) \wedge \phi^l + \sum_l \Big(\sum_k A_{kl}\pi^*\omega^i_k\Big) \wedge \phi^l \\
&= \sum_l dA_{il} \wedge \phi^l - \sum_{k,l} A_{ik}\pi^*\omega^k_l \wedge \phi^l + \sum_{k,l} A_{kl}\pi^*\omega^i_k \wedge \phi^l \\
&= \sum_l dA_{il} \wedge \phi^l - \sum_{k,l} A_{ik}\pi^*\omega^k_l \wedge \pi^*\theta^l + \sum_{k,l} A_{kl}\pi^*\omega^i_k \wedge \phi^l \\
&= \sum_l dA_{il} \wedge \phi^l + \sum_k A_{ik}\pi^*\Big(-\sum_l \omega^k_l \wedge \theta^l\Big) + \sum_{k,l} A_{kl}\pi^*\omega^i_k \wedge \phi^l.
\end{aligned}
$$

Using the first structure equation here, we obtain

$$
\begin{aligned}
\sum_{k,l} A_{il;k}\phi^k \wedge \phi^l &= \sum_k dA_{ik} \wedge \phi^k + \sum_k A_{ik}\pi^*(d\theta^k) + \sum_{k,l} A_{kl}\pi^*\omega^i_k \wedge \phi^l \\
&= \sum_k dA_{ik} \wedge \phi^k + \sum_k A_{ik}d(\pi^*\theta^k) + \sum_{k,l} A_{kl}\pi^*\omega^i_k \wedge \phi^l \\
&= \sum_k dA_{ik} \wedge \phi^k + \sum_k A_{ik}d(\phi^k) + \sum_{k,l} A_{kl}\pi^*\omega^i_k \wedge \phi^l.
\end{aligned}
$$

Then, replacing this last result in the expression obtained of Ψ^i_0, we arrive to

$$
\begin{aligned}
\Psi^i_0 &= -ab\sum_{k,l} A_{il;k}\phi^k \wedge \phi^l + \sum_{k,l}(ab)^2 A_{ik}A_{kl}\phi^0 \wedge \phi^l \\
&= -ab\sum_{k,l} A_{il;k}\phi^k \wedge \phi^l - a^2b^2\sum_{k,l} A_{ik}A_{kl}\phi^l \wedge \phi^0.
\end{aligned}
$$

For Ψ^i_j, using Theorem 3.1.8, we have

$$
\begin{aligned}
\Psi^i_j &= d\psi^i_j + \sum_a \psi^i_a \wedge \psi^a_j \\
&= d(\pi^*\omega^i_j - abA_{ij}\phi^0) + \sum_a \psi^i_a \wedge \psi^a_j \\
&= \pi^*(d\omega^i_j) - abdA_{ij} \wedge \phi^0 - abA_{ij}d\phi^0 + \psi^i_0 \wedge \psi^0_j + \sum_k \psi^i_k \wedge \psi^k_j \\
&= \pi^*(d\omega^i_j) - abdA_{ij} \wedge \phi^0 - abA_{ij}ab\Gamma + \Big(-\sum_k abA_{ik}\phi^k\Big) \wedge \Big(\sum_l abA_{jl}\phi^l\Big) + \\
&\quad \sum_k \Big((\pi^*\omega^i_k - abA_{ik}\phi^0) \wedge (\pi^*\omega^k_j - abA_{kj}\phi^0)\Big).
\end{aligned}
$$

Because of Γ is expressed as $\pi^*\Big(\sum_{k,l} A_{k,l}\theta^k \wedge \theta^l\Big)$, we obtain

$$
\begin{aligned}
\Psi^i_j = \pi^*(d\omega^i_j) - abdA_{ij} \wedge \phi^0 - (ab)^2 A_{ij}\pi^*\Big(\sum_{k,l} A_{k,l}\theta^k \wedge \theta^l\Big) - \\
\sum_{k,l}(ab)^2 A_{ik}A_{jl}\phi^k \wedge \phi^l + \sum_k \pi^*(\omega^i_k) \wedge \pi^*(\omega^k_j) - \\
\sum_k abA_{ik}\phi^0 \wedge \pi^*(\omega^k_j) - \sum_k abA_{kj}\pi^*(\omega^i_k) \wedge \phi^0.
\end{aligned}
$$

Putting together the expressions $\pi^*(d\omega^i_j)$ and $\sum_k \pi^*(\omega^i_k) \wedge \pi^*(\omega^k_j)$, we get

$$
\begin{aligned}
\Psi^i_j = \pi^*\Big(d\omega^i_j + \sum_k \omega^i_k \wedge \omega^k_j\Big) - (ab)^2\sum_{k,l}(A_{ij}A_{kl} + A_{ik}A_{jl})\phi^k \wedge \phi^l - \\
abdA_{ij} \wedge \phi^0 - \sum_k abA_{ik}\phi^0 \wedge \pi^*(\omega^k_j) - \sum_k abA_{kj}\pi^*\omega^i_k \wedge \phi^0.
\end{aligned}
$$

Then, by the second structure equation, that is $\Omega^i_j = d\omega^i_j + \sum_k \omega^i_k \wedge \omega^k_j$, we arrive to

$$\Psi^i_j = \pi^*(\Omega^i_j) - \sum_{k,l} a^2b^2(A_{ij}A_{kl} + A_{ik}A_{jl})\phi^k \wedge \phi^l -$$
$$ab\left(dA_{ij} - \sum_k A_{ik}\pi^*\omega^k_j + \sum_k A_{kj}\pi^*\omega^i_k\right) \wedge \phi^0.$$

Because of $\pi^*\left(dA_{ij} - \sum_k A_{ik}\omega^k_j + \sum_k A_{kj}\omega^i_k\right) = \sum_k A_{ij;k}\phi^k$, we conclude

$$\Psi^i_j = \pi^*(\Omega^i_j) - \sum_{k,l} a^2b^2(A_{ij}A_{kl} + A_{ik}A_{jl})\phi^k \wedge \phi^l - ab\sum_k A_{ij;k}\phi^k \wedge \phi^0.$$

□

Remark 3.1.10. We notice in the above theorem that $(A_{ij;\,k})$ consists in the covariant derivative of (1,1)-tensor $A = (A_{ij})$ with respect to the Riemannian connection of M. For this we need to verify that

$$\theta^i((\nabla_{E^k}A)E^j) = A_{ij;\,k}.$$

By a direct calculation we have

$$\begin{aligned}
(\nabla_{E^k}A)E^j &= \nabla_{E^k}(AE^j) - A(\nabla_{E^k}E^j)\\
&= \nabla_{E^k}\Big(\sum_l A_{lj}E^l\Big) - A\Big(\sum_l \omega^l_j(E^k)E^l\Big)\\
&= \sum_l (E^k(A_{lj})E^l + A_{lj}\nabla_{E^k}E^l) - \sum_l \omega^l_j(E^k)A(E^l)\\
&= \sum_l (E^k(A_{lj})E^l + A_{lj}\nabla_{E^k}E^l) - \sum_l \omega^l_j(E^k)\sum_m A_{ml}E^m\\
&= \sum_l (E^k(A_{lj})E^l + A_{lj}\nabla_{E^k}E^l) - \sum_l A_{ml}\omega^l_j(E^k)E^m.
\end{aligned}$$

Then, applying θ^i we get

$$\begin{aligned}
\theta^i((\nabla_{E^k}A)E^j) &= E^k(A_{ij}) + \sum_l A_{lj}\theta^i(\nabla_{E^k}E^l) - \sum_l A_{il}\omega^l_j(E^k)\\
&= dA_{ij}(E^k) + \sum_l A_{lj}\omega^i_l(E^k) - \sum_l A_{il}\omega^l_j(E^k)\\
&= A_{ij;\,k}.
\end{aligned}$$

We write the components $R_{\alpha\beta\lambda\mu}$ of the curvature tensor defined on P in Theorem 3.1.9 as

$$\Psi^{\alpha}_{\beta} = \frac{1}{2}\sum_{\lambda,\mu} R_{\alpha\beta\lambda\mu}\phi^{\lambda} \wedge \phi^{\mu}.$$

Proposition 3.1.11. *The components $R_{\alpha\beta\lambda\mu}$ are expressed in terms of K_{ijkl} and A_{ij} as*

(1) $R_{ijkl} = K_{ijkl} - a^2b^2(2A_{ij}A_{kl} + A_{ik}A_{jl} - A_{il}A_{jk})$,

(2) $R_{i0k0} = a^2b^2 \sum_l A_{il}A_{kl}$,

(3) $R_{i0kl} = ab(A_{ik;l} - A_{il;k}) = -abA_{kl;i}$.

Proof. We begin by showing item (2). By Theorem 3.1.9, we have

$$\Psi^{i}_{0} = \sum_{k,l} -a^2b^2A_{ik}A_{kl}\phi^{l} \wedge \phi^{0} - ab\sum_{k,l} A_{ik;l}\phi^{l} \wedge \phi^{k},$$

Then, comparing with the formula

$$\Psi^{i}_{0} = \frac{1}{2}\sum_{\lambda,\mu} R_{i0\lambda\mu}\phi^{\lambda} \wedge \phi^{\mu},$$

we get

$$\frac{1}{2}R_{i0l0} - \frac{1}{2}R_{i00l} = \sum_{k} -a^2b^2A_{ik}A_{kl},$$

and we arrive to

$$R_{i0l0} = -\sum_{k} a^2b^2A_{ik}A_{kl} = \sum_{k} a^2b^2A_{ik}A_{lk}.$$

For item (3), from

$$\frac{1}{2}R_{i0lk} - \frac{1}{2}R_{i0kl} = -abA_{ik;l} + abA_{il;k}.$$

It is trivial to obtain the first equality of (3).

For the last equality of (3), we start from $R_{i0kl} = R_{kli0}$. Considering the expression obtained in Theorem 3.1.9 for Ψ^{i}_{j}, that is

$$\Psi^{i}_{j} = \pi^{*}(\Omega^{i}_{j}) - \sum_{k,l} a^2b^2(A_{ij}A_{kl} + A_{ik}A_{jl})\phi^{k} \wedge \phi^{l} - ab\sum_{k} A_{ij;k}\phi^{k} \wedge \phi^{0},$$

we have $\frac{1}{2}R_{ijk0} = -abA_{ij;k}$ and $\frac{1}{2}R_{ij0k} = abA_{ij;k}$. Subtracting we get

$$\frac{1}{2}R_{ijk0} - \frac{1}{2}R_{ij0k} = -abA_{ij;k}.$$

As $R_{ij0k} = -R_{ijk0}$, we achieve $R_{ijk0} = -abA_{ij;k}$, our goal.

To show (1) we use again Theorem 3.1.9. Since $\Omega^i_j = \frac{1}{2}\sum_{k,l} K_{ijkl}\theta^k \wedge \theta^l$, we have

$$\begin{aligned}
\Psi^i_j &= \pi^*\Big(\frac{1}{2}\sum_{k,l} K_{ijkl}\theta^k \wedge \theta^l\Big) - \sum_{k,l} a^2b^2(A_{ij}A_{kl} + A_{ik}A_{jl})\phi^k \wedge \phi^l - \\
&\qquad ab\sum_k A_{ij;k}\,\phi^k \wedge \phi^0 \\
&= \frac{1}{2}\sum_{k,l} K_{ijkl}\phi^k \wedge \phi^l - \sum_{k,l} a^2b^2(A_{ij}A_{kl} + A_{ik}A_{jl})\phi^k \wedge \phi^l - ab\sum_k A_{ij;k}\,\phi^k \wedge \phi^0 \\
&= \sum_{k,l}\Big(\frac{1}{2}K_{ijkl} - a^2b^2(A_{ij}A_{kl} + A_{ik}A_{jl})\Big)\phi^k \wedge \phi^l - ab\sum_k A_{ij;k}\,\phi^k \wedge \phi^0.
\end{aligned}$$

So we get

$$\frac{1}{2}R_{ijkl} = \frac{1}{2}K_{ijkl} - a^2b^2(A_{ij}A_{kl} + A_{ik}A_{jl})$$

and

$$\frac{1}{2}R_{ijlk} = \frac{1}{2}K_{ijlk} - a^2b^2(A_{ij}A_{lk} + A_{il}A_{jk}).$$

Thus, we obtain

$$\frac{1}{2}R_{ijkl} - \frac{1}{2}R_{ijlk} = \frac{1}{2}K_{ijkl} - a^2b^2(A_{ij}A_{kl} + A_{ik}A_{jl}) - \frac{1}{2}K_{ijlk} + a^2b^2(A_{ij}A_{lk} + A_{il}A_{jk}).$$

From $R_{ijkl} = \frac{1}{2}R_{ijkl} - \frac{1}{2}R_{ijlk}$, we conclude the equality

$$\begin{aligned}
R_{ijkl} &= K_{ijkl} - a^2b^2(A_{ij}A_{kl} + A_{ik}A_{jl} - A_{ij}A_{lk} - A_{il}A_{jk}) \\
&= K_{ijkl} - a^2b^2(2A_{ij}A_{kl} + A_{ik}A_{jl} - A_{il}A_{jk}).
\end{aligned}$$

□

As consequence we deduce that the coefficients $A_{ij;k}$ also satisfy the following conditions:

i) $A_{ij;k} = -A_{ji;k}$,

ii) $A_{ij;k} + A_{jk;i} + A_{ki;j} = 0$.

3.2 Algebraic properties of the sectional curvature

Calculations in this section are taken from Kobayashi [18]. Let M be a smooth manifold with even real dimension n. Let K_{ijkl} be a collection of real numbers that verify

$$K_{ijkl} = -K_{jikl} = -K_{ijlk} = K_{klij}, \tag{3.9}$$

$$K_{ijkl} + K_{iklj} + K_{iljk} = 0, \tag{3.10}$$

the conditions of the components of a curvature tensor.

Consider the skew-symmetric matrix $J = (J_{ij})$ with $J^2 = -I_n$, and a collection of real numbers $S_{\alpha\beta\lambda\mu}$ characterized by:

$$\begin{aligned}
S_{ijkl} &= K_{ijkl} - a^2(2J_{ij}J_{kl} + J_{ik}J_{jl} - J_{il}J_{jk}),\\
S_{i0k0} &= -S_{i00k} = -S_{0ik0} = S_{0i0k} = a^2\delta_{ik},\\
S_{\alpha\beta\lambda\mu} &= 0, \quad \text{otherwise.}
\end{aligned}$$

We will show that $S_{\alpha\beta\lambda\mu}$ satisfy the algebraic conditions of the components of the curvature tensor (i.e. $S_{\alpha\beta\lambda\mu}$ verifies also conditions (3.9) and (3.10)). In fact, we have

$$\begin{aligned}
S_{ijlk} &= K_{ijlk} - a^2(2J_{ij}J_{lk} + J_{il}J_{jk} - J_{ik}J_{jl})\\
&= -K_{ijkl} - a^2(-2J_{ij}J_{kl} - J_{ik}J_{jl} + J_{il}J_{jk})\\
&= -(K_{ijkl} - a^2(2J_{ij}J_{kl} + J_{ik}J_{jl} - J_{il}J_{jk}))\\
&= -S_{ijkl}.
\end{aligned}$$

It is also easy to prove that $S_{jikl} = S_{ijlk}$ holds. Moreover, we have

$$\begin{aligned}
S_{klij} &= K_{klij} - a^2(2J_{kl}J_{ij} + J_{ki}J_{lj} - J_{kj}J_{li})\\
&= K_{ijkl} - a^2(2J_{ij}J_{kl} + J_{ik}J_{jl} - J_{jk}J_{il})\\
&= S_{ijkl}.
\end{aligned}$$

Finally, we observe that S_{ijkl} also satisfies

$$S_{ijkl} + S_{iklj} + S_{iljk} = 0.$$

Indeed, by definition we have

$$\begin{aligned}
S_{ijkl} &= K_{ijkl} - a^2(2J_{ij}J_{kl} + J_{ik}J_{jl} - J_{il}J_{jk}),\\
S_{iklj} &= K_{iklj} - a^2(2J_{ik}J_{lj} + J_{il}J_{kj} - J_{ij}J_{kl}) \text{ and}\\
S_{iljk} &= K_{iljk} - a^2(2J_{il}J_{jk} + J_{ij}J_{lk} - J_{ik}J_{lj}).
\end{aligned}$$

Adding the above equalities, we obtain the result required.

Remark 3.2.1. Supposing K_{ijkl} were the components of a curvature tensor K of the metric ds^2, then we would have that $S_{\alpha\beta\lambda\mu}$ should be the components of the curvature tensor of the metric $d\sigma^2 = \pi^* ds^2 + (ab\gamma)^2$ on P when the curvature form Γ is expressed as

$$\Gamma = d\gamma = \pi^*\Big(\sum_{i,j=1}^{n} b^{-1} J_{ij}\theta^i \wedge \theta^j\Big),$$

with $A_{ij;k} = 0$. By Remark 3.1.10, this means that the tensor $J = J_{ij}$ is parallel with respect to metric ds^2.

For each 2-dimensional subspace $p \subset \mathrm{R}^{n+1}$, we define a number $S(p)$ as follows. Fix $\{X, Y\} \subset p$ an orthonormal basis. Then we write

$$S(p) = \sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\lambda\mu} X^\alpha Y^\beta X^\lambda Y^\mu,$$

where $X = (X^0, X^1, ..., X^n)$ and $Y = (Y^0, Y^1, ..., Y^n)$.

Let us see why the number $S(p)$ is independent of the basis $\{X, Y\} \subset p$. Take another orthonormal basis $\{\tilde{X}, \tilde{Y}\} \subset p$. Then we can express this basis as

$$\tilde{X} = a_{11}X + a_{21}Y$$
$$\tilde{Y} = a_{12}X + a_{22}Y.$$

Since the matrix $(a_{ij})_{2\times 2}$ is orthonormal, we must have $a_{11}^2 + a_{21}^2 = 1$. Moreover, these number verify precisely one of the following conditions

$$a_{11} = a_{22} \text{ and } a_{12} = -a_{21} \quad \text{or} \quad a_{11} = -a_{22} \text{ and } a_{12} = a_{21}.$$

Supposing $a_{11} = a_{22}$ and $a_{12} = -a_{21}$, we can express $S(p)$ as

$$\begin{aligned} S(p) &= \sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\lambda\mu} \tilde{X}^\alpha \tilde{Y}^\beta \tilde{X}^\lambda \tilde{Y}^\mu \\ &= \sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\lambda\mu}(a_{11}X^\alpha - a_{12}Y^\alpha)(a_{12}X^\beta + a_{11}Y^\beta)(a_{11}X^\lambda - a_{12}Y^\lambda)(a_{12}X^\mu + a_{11}Y^\mu). \end{aligned}$$

Notice that the terms $\sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\lambda\mu} X^\alpha X^\beta X^\lambda Y^\mu$ and $\sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\lambda\mu} X^\alpha Y^\beta Y^\lambda Y^\mu$ are

0. Indeed, we have

$$
\begin{aligned}
\sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\lambda\mu} X^\alpha X^\beta X^\lambda Y^\mu &= \sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\lambda\mu} X^\beta X^\alpha X^\lambda Y^\mu \\
&= \sum_{\alpha,\beta,\lambda,\mu} S_{\beta\alpha\lambda\mu} X^\alpha X^\beta X^\lambda Y^\mu \\
&= -\sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\lambda\mu} X^\alpha X^\beta X^\lambda Y^\mu,
\end{aligned}
$$

which implies $\sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\lambda\mu} X^\alpha X^\beta X^\lambda Y^\mu = 0$. In a similar way we obtain $\sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\lambda\mu} X^\alpha Y^\beta Y^\lambda Y^\mu = 0$.

Therefore, we can reduce $S(p)$ to

$$
\begin{aligned}
S(p) = & \sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\lambda\mu}(a_{11}^4 X^\alpha Y^\beta X^\lambda Y^\mu - a_{11}^2 a_{12}^2 (X^\alpha Y^\beta Y^\lambda X^\mu + Y^\alpha X^\beta X^\lambda Y^\mu) + \\
& a_{12}^4 Y^\alpha X^\beta Y^\lambda X^\mu) \\
= & \sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\lambda\mu} a_{11}^4 X^\alpha Y^\beta X^\lambda Y^\mu - \sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\lambda\mu} a_{11}^2 a_{12}^2 X^\alpha Y^\beta Y^\lambda X^\mu - \\
& \sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\lambda\mu} a_{12}^2 a_{11}^2 Y^\alpha X^\beta X^\lambda Y^\mu + \sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\lambda\mu} a_{12}^4 Y^\alpha X^\beta Y^\lambda X^\mu \\
= & \sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\lambda\mu} a_{11}^4 X^\alpha Y^\beta X^\lambda Y^\mu - \sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\mu\lambda} a_{11}^2 a_{12}^2 X^\alpha Y^\beta X^\lambda Y^\mu - \\
& \sum_{\alpha,\beta,\lambda,\mu} S_{\beta\alpha\lambda\mu} a_{12}^2 a_{11}^2 X^\alpha Y^\beta X^\lambda Y^\mu + \sum_{\beta,\alpha,\mu,\lambda} S_{\alpha\beta\lambda\mu} a_{12}^4 X^\alpha Y^\beta X^\lambda Y^\mu \\
= & \sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\lambda\mu}(a_{11}^4 + 2a_{11}^2 a_{12}^2 + a_{12}^4) X^\alpha Y^\beta X^\lambda Y^\mu.
\end{aligned}
$$

As $a_{11}^2 + a_{12}^2 = 1$, we get

$$
S(p) = \sum_{\alpha,\beta,\lambda,\mu} S_{\alpha\beta\lambda\mu} X^\alpha Y^\beta X^\lambda Y^\mu.
$$

Thus, the number $S(p)$ is independent of the choice of orthonormal frame.

Given orthonormal basis $X = (X^0, X^1, \ldots, X^n)$, $Y = (Y^0, Y^1, \ldots, Y^n)$ of a 2-plane $p \subset \mathbb{R}^{n+1}$, we define $\xi = (X^1, \ldots, X^n)$ and $\eta = (Y^1, \ldots, Y^n)$ in $\mathbb{R}^n$. Then we have

$$
J\xi = \Big(\sum_j J_{1j} X^j, \ldots, \sum_j J_{nj} X^j\Big), \tag{3.11}
$$

$$
J\eta = \Big(\sum_j J_{1j} Y^j, \ldots, \sum_j J_{nj} Y^j\Big). \tag{3.12}
$$

In the following proposition we express $S(p)$ in terms of η and ξ.

Proposition 3.2.2. *Let $p \subset \mathbb{R}^{n+1}$ as above. Then*

$$S(p) = \sum_{i,j,k,l} K_{ijkl} X^i Y^j X^k Y^l - 3a^2(\xi, J\eta)^2 + a^2(X^0X^0 + Y^0Y^0).$$

Proof. Expanding $S(p)$, we have

$$S(p) = \sum_{i,j,k,l} S_{ijkl} X^i Y^j X^k Y^l + \sum_{i,k} S_{i0k0} X^i Y^0 X^k Y^0 + \sum_{i,l} S_{i00l} X^i Y^0 X^0 Y^l + \sum_{j,l} S_{0j0l} X^0 Y^j X^0 Y^l + \sum_{j,k} S_{0jk0} X^0 Y^j X^k Y^0.$$

We work individually on each summand of the above expression. For the first term we obtain

$$\begin{aligned}\sum_{i,j,k,l} S_{ijkl} X^i Y^j X^k Y^l &= \sum_{i,j,k,l} (K_{ijkl} - a^2(2J_{ij}J_{kl} + J_{ik}J_{jl} - J_{il}J_{jk}))X^i Y^j X^k Y^l \\ &= \sum_{i,j,k,l} K_{ijkl} X^i Y^j X^k Y^l - 2a^2 \sum_{i,j,k,l} J_{ij}J_{kl} X^i Y^j X^k Y^l - \\ &\quad a^2 \sum_{i,j,k,l} J_{ik}J_{jl} X^i Y^j X^k Y^l + a^2 \sum_{i,j,k,l} J_{il}J_{jk} X^i Y^j X^k Y^l.\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{i,j,k,l} J_{ik}J_{jl} X^i Y^j X^k Y^l &= \sum_{i,j,k,l} J_{ik}J_{jl} X^k Y^j X^i Y^l \\ &= \sum_{i,j,k,l} J_{ki}J_{jl} X^i Y^j X^k Y^l \\ &= -\sum_{i,j,k,l} J_{ik}J_{jl} X^i Y^j X^k Y^l\end{aligned}$$

implies $\sum_{i,j,k,l} J_{ik}J_{jl} X^i Y^j X^k Y^l = 0$. In addition, we have

$$\sum_{i,j,k,l} J_{il}J_{jk} X^i Y^j X^k Y^l = \sum_{i,j,k,l} J_{ij}J_{kl} X^i Y^j X^k Y^l.$$

As a consequence, considering the last two results and Equation (3.12), we obtain

$$\begin{aligned}\sum_{i,j,k,l} S_{ijkl} X^i Y^j X^k Y^l &= \sum_{i,j,k,l} K_{ijkl} X^i Y^j X^k Y^l - 3a^2 \sum_{i,j,k,l} J_{ij}J_{kl} X^i Y^j X^k Y^l \\ &= \sum_{i,j,k,l} K_{ijkl} X^i Y^j X^k Y^l - 3a^2 \sum_{i,j,k,l} J_{ij} X^i Y^j \Big(\sum_{k,l} J_{kl} X^k Y^l\Big) \\ &= \sum_{i,j,k,l} K_{ijkl} X^i Y^j X^k Y^l - 3a^2(\xi, J\eta)^2.\end{aligned}$$

The other terms of $S(p)$ are

$$\begin{aligned}
\sum_{i,k} S_{i0k0}X^iY^0X^kY^0 &= \sum_{i,k} a^2\delta_{ik}X^iY^0X^kY^0 \\
&= a^2Y^0Y^0\sum_{i,k}\delta_{ik}X^iX^k \\
&= a^2Y^0Y^0\sum_{i} X^iX^i \\
&= a^2(\xi,\xi)Y^0Y^0,
\end{aligned}$$

$$\begin{aligned}
\sum_{i,l} S_{i00l}X^iY^0X^0Y^l &= -\sum_{i,l} S_{i0l0}X^iY^0X^0Y^l \\
&= -\sum_{i,l} a^2\delta_{il}X^iY^0X^0Y^l \\
&= -a^2Y^0X^0\sum_{i,l}\delta_{il}X^iY^l \\
&= -a^2Y^0X^0\sum_{i} X^iY^i \\
&= -a^2(\xi,\eta)X^0Y^0,
\end{aligned}$$

$$\begin{aligned}
\sum_{j,l} S_{0j0l}X^0Y^jX^0Y^l &= \sum_{j,l} a^2\delta_{jl}X^0Y^jX^0Y^l \\
&= a^2X^0X^0\sum_{j,l}\delta_{jl}Y^jY^l \\
&= a^2(\eta,\eta)X^0X^0,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j,k} S_{0jk0}X^0Y^jX^kY^0 &= -\sum_{j,k} a^2\delta_{jk}X^0Y^jX^kY^0 \\
&= -a^2X^0Y^0\sum_{j} Y^jX^j \\
&= -a^2(\xi,\eta)X^0Y^0.
\end{aligned}$$

Replacing them in the expression of $S(p)$, we arrive to

$$S(p) = \sum_{i,j,k,l} K_{ijkl}X^iY^jX^kY^l - 3a^2(\xi,J\eta)^2 + a^2(\xi,\xi)Y^0Y^0 -$$
$$2a^2(\xi,\eta)X^0Y^0 + a^2(\eta,\eta)X^0X^0.$$

However, using the equalities

$$\sum_{\alpha} X^{\alpha}X^{\alpha} = X^0X^0 + (\xi, \xi) = 1$$

$$\sum_{\beta} Y^{\beta}Y^{\beta} = Y^0Y^0 + (\eta, \eta) = 1$$

$$(X, Y) = X^0Y^0 + (\xi, \eta) = 0,$$

we have $(\xi, \xi) = 1 - X^0X^0$, $(\eta, \eta) = 1 - Y^0Y^0$, and $(\xi, \eta) = -X^0Y^0$. Replacing these results in $S(p)$, we get

$$\begin{aligned} S(p) &= \sum_{i,j,k,l} K_{ijkl}X^iY^jX^kY^l - 3a^2(\xi, J\eta)^2 + a^2(1 - X^0X^0)Y^0Y^0 - \\ &\quad 2a^2(-X^0Y^0)X^0Y^0 + a^2(1 - Y^0Y^0)X^0X^0 \\ &= \sum_{i,j,k,l} K_{ijkl}X^iY^jX^kY^l - 3a^2(\xi, J\eta)^2 + a^2(X^0X^0 + Y^0Y^0), \end{aligned}$$

as wished. □

Let K_{ijkl} be the collection of real numbers given at beginning of this section. For a 2-dimensional subspace $q \subset \mathbb{R}^n$, we define the numbers $K(q)$ and $\alpha(q)$ by

$$K(q) = \sum_{i,j,k,l} K_{ijkl}U^iV^jU^kV^l$$

and

$$\cos\alpha(q) = |(U, JV)|,$$

where $U = (U^1, ..., U^n)$, $V = (V^1, ..., V^n)$ form an orthonormal basis of q. We have already seen that $K(q)$ in independent of the choice of the basis $\{U, V\}$. Let us see that $\cos\alpha(q)$ also depends only on q. Indeed, if $\{\tilde{U}, \tilde{V}\}$ is another orthonormal basis for q, then we must verify the equality

$$|(U, JV)| = |(\tilde{U}, J\tilde{V})|.$$

Since we can write $\tilde{U}$ and $\tilde{V}$ as

$$\tilde{U} = a_{11}U + a_{21}V \text{ and } \tilde{V} = a_{12}U + a_{22}V,$$

we get then

$$\begin{aligned} (\tilde{U}, J\tilde{V}) &= (a_{11}U + a_{21}V, a_{12}JU + a_{22}JV) \\ &= a_{11}a_{12}(U, JU) + a_{21}a_{12}(V, JU) + a_{11}a_{22}(U, JV) + a_{21}a_{22}(V, JV). \end{aligned}$$

Since we have

$$(U, JU) = \sum_k U^k \Big(\sum_j J_{kj} U^j\Big) = \sum_{j,k} J_{kj} U^k U^j$$

and

$$\sum_{jk} J_{kj} U^k U^j = \sum_{jk} J_{jk} U^j U^k = -\sum_{jk} U^k U^j,$$

we obtain $(U, JU) = 0$. In a similar way we get $(V, JV) = 0$. On the other hand, for $(\tilde{U}, J\tilde{V})$, we have

$$\begin{aligned}
(\tilde{U}, J\tilde{V}) &= a_{21}a_{12}(V, JU) + a_{11}a_{22}(U, JV) \\
&= a_{21}a_{12} \sum_k V^k \sum_j J_{kj} U^j + a_{11}a_{22} \sum_k U^k \sum_j J_{kj} V^j \\
&= a_{21}a_{12} \sum_{j,k} J_{kj} U^j V^k + a_{11}a_{22} \sum_{j,k} J_{kj} U^k V^j \\
&= -a_{21}a_{12} \sum_{j,k} J_{jk} U^j V^k + a_{11}a_{22} \sum_{j,k} J_{jk} U^j V^k \\
&= (a_{11}a_{22} - a_{21}a_{12}) \sum_{j,k} J_{jk} U^j V^k \\
&= (a_{11}a_{22} - a_{21}a_{12})(U, JV).
\end{aligned}$$

Since $|a_{11}a_{22} - a_{21}a_{12}| = 1$, we conclude $|(\tilde{U}, J\tilde{V})| = |(U, JV,)|$, as needed.

Remark 3.2.3. The value $\alpha(q)$ measures the angle between the planes q and Jq. In fact, we recall that the angle between two planes p and p' is defined as the infimum $\alpha = \angle(A, A')$, where A is a vector in p and A' is a vector in p'. Then in the case of the planes q and Jq, we consider $\{U, V\}$ be a orthonormal basis of q and $\{JU, JV\}$ be a basis orthonormal of Jq. Fixing U in q, we should minimize the angle $\angle(U, X)$, where $X \in Jq$. Writing $X = t_1 JU + t_2 JV$, we have

$$\cos(\alpha) = \frac{|(U, t_1 JU + t_2 JV)|}{\|U\| \|t_1 JU + t_2 JV\|},$$

where α is the angle between U and X.

Since $(U, JU) = 0$ and $\{JU, JV\}$ is an orthonormal basis, we obtain

$$\cos(\alpha) = \frac{|(U, t_2 JV)|}{\|t_1 JU\| + \|t_2 JV\|} = \frac{|t_2| |(U, JV)|}{|t_1| + |t_2|}.$$

The angle α where $\cos(\alpha)$ takes its value maximum is when $t_1 = 0$. In other words, the angle $\alpha(q)$ between q and Jq is given by $\cos(\alpha(q)) = |(U, JV)|$.

Let $p = \operatorname{span}\{X, Y\}$, where $X = (X^0, \ldots, X^n)$ and $Y = (Y^0, \ldots, Y^n)$. We define the subspace $q \subset \mathbb{R}^n$ as $q = \operatorname{span}\{\xi, \eta\}$, where $\xi = (X^1, \ldots, X^n)$ and $\eta = (Y^1, \ldots, Y^n)$. For them we have the following proposition.

Proposition 3.2.4. *(1) For $p \subset \mathbb{R}^{n+1}$ we have*

$$S(p) = (1 - X^0X^0 - Y^0Y^0)(K(q) - 3a^2\cos^2\alpha(q)) + a^2(X^0X^0 + Y^0Y^0).$$

(2) If ξ, η are linearly dependent, then $S(p) = a^2$

Proof. To prove (1), we use the Gram-Schmidt process. An orthonormal basis $\{U, V\}$ for q is

$$U = \frac{\xi}{(\xi\xi)^{\frac{1}{2}}} \tag{3.13}$$

$$V = \frac{[(\xi,\xi)\eta - (\xi,\eta)\xi]}{[(\xi,\xi)((\xi,\xi)(\eta,\eta) - (\xi,\eta)^2)]^{\frac{1}{2}}}. \tag{3.14}$$

Then, for $\cos(\alpha(q))$, we obtain

$$\begin{aligned}\cos(\alpha(q)) = |\langle U, JV\rangle| &= \left\langle \frac{\xi}{(\xi,\xi)^{\frac{1}{2}}}, \frac{(\xi,\xi)J\eta - (\xi,\eta)J\xi}{[(\xi,\xi)((\xi,\xi)(\eta,\eta) - (\xi,\eta)^2)]^{\frac{1}{2}}} \right\rangle \\ &= \frac{(\xi, (\xi,\xi)J\eta - (\xi,\eta)J\xi)}{(\xi,\xi)((\xi,\xi)(\eta,\eta) - (\xi,\eta)^2)^{\frac{1}{2}}} \\ &= \frac{(\xi,\xi)(\xi,J\eta)}{(\xi,\xi)((\xi,\xi)(\eta,\eta) - (\xi,\eta)^2)^{\frac{1}{2}}} \\ &= \frac{(\xi,J\eta)}{((\xi,\xi)(\eta,\eta) - (\xi,\eta)^2)^{\frac{1}{2}}},\end{aligned}$$

and we have

$$(\xi, J\eta) = \cos(\alpha(q))((\xi,\xi)(\eta,\eta) - (\xi,\eta)^2)^{\frac{1}{2}}.$$

Squaring this last expression, we get

$$(\xi, J\eta)^2 = ((\xi,\xi)(\eta,\eta) - (\xi,\eta)^2)\cos^2(\alpha(q)). \tag{3.15}$$

On the other hand, by Equalities (3.13) and (3.14), we have

$$(\xi,\xi)^{\frac{1}{2}}U^i = X^i$$
$$[(\xi,\xi)((\xi,\xi)(\eta,\eta) - (\xi,\eta)^2)]^{\frac{1}{2}}V^j = (\xi,\xi)Y^j - (\xi,\eta)X^j.$$

Replacing the above formulas in $K(q) = \sum_{i,j,k,l} K_{ijkl} U^i V^j U^k V^l$, we get

$$
\begin{aligned}
K(q) &= \sum_{i,j,k,l} K_{ijkl} \frac{X^i((\xi,\xi)Y^j - (\xi,\eta)X^j)X^k((\xi,\xi)Y^l - (\xi,\eta)X^l)}{(\xi,\xi)[(\xi,\xi)((\xi,\xi)(\eta,\eta) - (\xi,\eta)^2)]} \\
&= \sum_{i,j,k,l} K_{ijkl} \frac{(\xi,\xi)^2 X^i Y^j X^k Y^l}{(\xi,\xi)[(\xi,\xi)((\xi,\xi)(\eta,\eta) - (\xi,\eta)^2)]} \\
&= \sum_{i,j,k,l} K_{ijkl} \frac{X^i Y^j X^k Y^l}{(\eta,\eta) - (\xi,\eta)^2}.
\end{aligned}
$$

Hence we have

$$\sum_{i,j,k,l} K_{ijkl} X^i Y^j X^k Y^l = K(q)[(\eta,\eta) - (\xi,\eta)^2].$$

But

$$
\begin{aligned}
(\eta,\eta) - (\xi,\eta)^2 &= (1 - X^0X^0)(1 - Y^0Y^0) - (X^0X^0Y^0Y^0) \\
&= 1 - X^0X^0 - Y^0Y^0
\end{aligned}
$$

implies

$$\sum_{i,j,k,l} K_{ijkl} X^i Y^j X^k Y^l = K(q)(1 - X^0X^0 - Y^0Y^0). \tag{3.16}$$

Replacing Formulas (3.15) and (3.16) in the expression for $S(p)$ obtained in Proposition 3.2.2, we get

$$
\begin{aligned}
S(p) = &\sum_{i,j,k,l} K_{ijkl} X^i Y^j X^k Y^l - 3a^2(\xi, J\eta)^2 + a^2(X^0X^0 + Y^0Y^0) \\
&= (1 - X^0X^0 - Y^0Y^0)K(q) - 3a^2(1 - X^0X^0 - Y^0Y^0)\cos^2\alpha(q) + \\
&\quad a^2(X^0X^0 + Y^0Y^0) \\
&= (1 - X^0X^0 - Y^0Y^0)(K(q) - 3a^2\cos^2\alpha(q)) + a^2(X^0X^0 + Y^0Y^0).
\end{aligned}
$$

If $\{\xi,\eta\}$ were linearly dependent, then $\eta = c\xi$. This means $Y^j = cX^j$, where c is a constant, and we get

$$\sum_{i,j,k,l} K_{ijkl} X^i Y^j X^k Y^l = c^2 \sum_{i,j,k,l} K_{ijkl} X^i X^j X^k X^l = 0,$$

and $(\xi, J\eta) = c(\xi, J\xi) = 0$. Moreover, since

$$(\xi,\xi)(\eta,\eta) - (\xi,\eta)^2 = 1 - X^0X^0 - Y^0Y^0,$$

we obtain

$$0 = c^2(\xi,\xi)^2 - c^2(\xi,\xi)^2 = 1 - X^0X^0 - Y^0Y^0.$$

Therefore $X^0X^0 + Y^0Y^0 = 1$, and we get $S(p) = a^2$. □

To simplify the exposition we introduce the quantity

$$\bar{K}(q) = \frac{1 + 3\cos^2\alpha(q)}{4}.$$

Proposition 3.2.5. *If a is a real number such that $a \leq \frac{1}{2}$, and $q = span\{\xi, \eta\}$, verifies*

$$4a^2\bar{K}(q) \leq K(q) \leq \bar{K}(q),$$

then we get

$$a^2 \leq S(p) \leq 1 - 3a^2.$$

Proof. First, we find the lower bound for $S(p)$. By hypothesis we have

$$K(q) \geq 4a^2\bar{K}(q) = a^2(1 + 3\cos^2(\alpha(q))),$$

and so $K(q) - 3a^2\cos^2\alpha(q) \geq a^2$.
So, because of Proposition 3.2.4., we get

$$\begin{aligned} S(p) &= (1 - X^0X^0 - Y^0Y^0)(K(q) - 3a^2\cos^2\alpha(q)) + a^2(X^0X^0 + Y^0Y^0) \\ &\geq (1 - X^0X^0 - Y^0Y^0)a^2 + a^2(X^0X^0 + Y^0Y^0) \\ &= a^2 \end{aligned}$$

For an upper bound for $S(p)$ we start from

$$\begin{aligned} K(q) - 3a^2\cos^2\alpha(q) &\leq \bar{K}(q) - 3a^2\cos^2\alpha(q) \\ &= \frac{1}{4}(1 + 3\cos^2\alpha(q)) - 3a^2\cos^2\alpha(q) \\ &= \frac{1}{4}(1 + 3(1 - 4a^2)\cos^2\alpha(q)) \\ &\leq \frac{1}{4}(1 + 3(1 - 4a^2)) \\ &= 1 - 3a^2 \end{aligned}$$

So for $S(p)$ we obtain

$$\begin{aligned} S(p) &= (1 - X^0X^0 - Y^0Y^0)(K(q) - 3a^2\cos^2\alpha(q)) + a^2(X^0X^0 + Y^0Y^0) \\ &\leq (1 - X^0X^0 - Y^0Y^0)(1 - 3a^2) + a^2(X^0X^0 + Y^0Y^0) \\ &= 1 - 3a^2 + (4a^2 - 1)(X^0X^0 + Y^0Y^0). \end{aligned}$$

Since $4a^2 - 1 \leq 0$, this results in

$$S(p) \leq 1 - 3a^2,$$

as claimed. □

Consider now real numbers A_{ij} and $A_{ij;k}$ related by

(1) $A_{ij} = -A_{ji}$.

(2) $A_{ij;k} = -A_{ji;k}$.

(3) $A_{ij;k} + A_{ki;j} + A_{jk;i} = 0$.

For them we define $R_{\alpha\beta\lambda\mu}$ as

(1) $R_{ijkl} = K_{ijkl} - a^2b^2(2A_{ij}A_{kl} + A_{ik}A_{jl} - A_{il}A_{jk})$,

(2) $R_{i0k0} = a^2b^2 \sum_l A_{il}A_{kl}$,

(3) $R_{i0kl} = ab(A_{ik;l} - A_{il;k}) = -abA_{kl;i}$.

Supposing $\{X, Y\}$ is an orthonormal basis for a 2-subspace $p \subset \mathbb{R}^{n+1}$, we define

$$R(p) = \sum_{\alpha\beta\lambda\mu} R_{\alpha\beta\lambda\mu} X^\alpha Y^\beta X^\lambda Y^\mu.$$

The following proposition shows us how the numbers $R(p)$ and $S(p)$ approach while A_{ij} nears J_{ij}.

Proposition 3.2.6. *Let a be a fixed real positive number. Given $s > 0$, then there exists a number $\rho > 0$ such that if $\sum_{ij} |bA_{ij} - J_{ij}|^2 < \rho$ and $\sum_{ijk} |bA_{ij;k}|^2 < \rho$, we get $|R(p) - S(p)| < s$.*

Proof. By definition of R_{ijkl} and S_{ijkl}, if $bA_{ij} \to J_{ij}$ and $bA_{ij;k} \to 0$, we get

$$R_{ijkl} \to K_{ijkl} - a^2(2J_{ij}J_{kl} + J_{ik}J_{jl} - J_{il}J_{jk}) = S_{ijkl}.$$

On the other hand, as $bA_{ij} \to J_{ij}$, we get

$$R_{i0k0} = a^2b^2 \sum_l A_{il}A_{kl} \to a^2 \sum_l J_{il}J_{kl}.$$

Also we have

$$a^2 \sum_l J_{il}J_{kl} = -a^2 \sum_l J_{il}J_{lk} = -a^2(-\delta_{ik}) = a^2\delta_{ik} = S_{i0k0},$$

and so $R_{i0k0} \to S_{i0k0}$.

To finish, we notice that $bA_{ij;k} \to 0$, implies $R_{i0kl} \to 0 = S_{i0kl}$. □

3.3 Construction of a principal S^1-bundle

Let (M, g, J) be a complete Kähler manifold and q a real 2-plane in T_xM. Taking $\{X, Y\}$ as a basis for q, we define the *sectional curvature* of q as

$$K(q) = \frac{K(X, Y, X, Y)}{g(X, X)g(Y, Y) - (g(X, Y))^2}.$$

When $\{X, Y\}$ is an orthonormal basis of q, the above formula results in

$$K(q) = K(X, Y, X, Y).$$

As usual, for a 2-plane q we define the *angle between q and Jq* by $\alpha(q)$. In a similar way as was done in Remark 3.2.3., we deduce the equality

$$\cos \alpha(q) = g(X, JY),$$

whenever $\{X, Y\}$ is an orthonormal basis for q.

For the 2-plane q we define its *Kählerian sectional curvature*

$$K^*(q) = \frac{4K(q)}{1 + 3\cos^2 \alpha(q)}.$$

Definition 3.3.1. Let (M, g, J) be a Kähler manifold. We say that the *Riemannian pinching* of M is greater than δ if there is $L > 0$ such that

$$\delta L < K(q) \leq L \qquad \text{for all 2- planes } q.$$

Definition 3.3.2. Let (M, g, J) be a Kähler manifold. We say that the *Kählerian pinching* of M is greater than δ if there is $L > 0$ such that

$$\delta L < K^*(q) \leq L \qquad \text{for all 2-planes } q.$$

We are ready to state the main theorem of this chapter.

Theorem 3.3.3. *Let (M, J, g) be a complete Kähler manifold with Kählerian pinching greater than δ. Then there exists a principal circle bundle P over M and a Riemannian metric on P with Riemannian pinching greater than $\dfrac{\delta}{(4 - 3\delta)}$.*

The proof of this theorem aims for a suitable principal S^1-bundle P on M so that we can define a connection form γ on P with the sectional curvature induced by the metric $d\sigma^2 = ds^2 + (ab\gamma)^2$ that verifies the condition of theorem.

First, we show the existence of this principal S^1-bundle.

By de-Rham's Theorem, we have $H^2(M, \mathbb{R}) \cong H^2_{dR}(M, \mathbb{R})$. Since the constant sheaf $\mathbb{Z}$ is included in $\mathbb{R}$, we obtain a homomorphism $\varphi : H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$. As $H^2(M, \mathbb{Z}) \cong \mathcal{P}(M, S^1)$, then we can assign to each element $P \in \mathcal{P}(M, S^1)$ an element in $H^2_{dR}(M, \mathbb{R})$ (see Kobayashi [19]).

This assignment can be made explicit as follows. Let P an element of $\mathcal{P}(M, S^1)$ and γ a connection form on P. By Proposition 2.1.16, we can express $\Gamma = d\gamma$ as

$$\Gamma = d\gamma = \pi^*\Big(\sum_{i,j} B_{ij}\theta^i \wedge \theta^j\Big).$$

We have them

$$0 = d^2\gamma = d\Gamma = \pi^* d\Big(\sum_{i,j} B_{ij}\theta^i \wedge \theta^j\Big).$$

As π^* is injective, so $d\Big(\sum_{i,j} B_{ij}\theta^i \wedge \theta^j\Big) = 0$. Then $\sum_{i,j} B_{ij}\theta^i \wedge \theta^j$ is closed.

The de-Rham cohomology class of $\sum_{i,j} B_{ij}\theta^i \wedge \theta^j$ is independent of the choice of a connection form on P. Indeed, by Proposition 2.1.17, if we take another connection form $\tilde{\gamma}$ on P, we have $\gamma - \tilde{\gamma} = \pi^*\tau$, where τ is a 1-form on M. In this way we get

$$\pi^*(d\tau) = d\gamma - d\tilde{\gamma} = \Gamma - \tilde{\Gamma} = \pi^*\Big(\sum_{i,j} B_{ij}\theta^i \wedge \theta^j - \sum_{i,j} \tilde{B}_{ij}\theta^i \wedge \theta^j\Big).$$

Since π^* is injective, we obtain

$$d\tau = \sum_{i,j} B_{ij}\theta^i \wedge \theta^j - \sum_{i,j} \tilde{B}_{ij}\theta^i \wedge \theta^j.$$

Therefore our statement is true.

Finally based on the work of Kobayashi [19], we can choose $\sum_{i,j} B_{ij}\theta^i \wedge \theta^j$ so that it is an element of $H^2_{dR}(M, \mathbb{Z})$. The converse is also studied in [19].

In the following proposition we show that given any 2-form $\sum_{i,j} A_{ij}\theta^i \wedge \theta^j$ in $H^2_{dR}(M, \mathbb{Z})$, we can express its projection $\pi^*\big(\sum_{i,j} A_{ij}\theta^i \wedge \theta^j\big)$ as $d\gamma$, where γ is a connection form on P.

Proposition 3.3.4. *Given a 2-form $\sum_{i,j} A_{ij}\theta^i \wedge \theta^j$ representing an element of $H^2_{DR}(M,\mathbb{Z})$, there is a principal circle bundle P and a connection form γ on P such that*

$$d\gamma = \pi^*\Big(\sum_{i,j} A_{ij}\theta^i \wedge \theta^j\Big).$$

Proof. Let $\sum_{i,j} A_{ij}\theta^i \wedge \theta^j$ be a 2-form representing an element of $H^2_{DR}(M,\mathbb{Z})$. By the isomorphism $\mathsf{P}(M,S^1) \cong H^2(M,\mathbb{Z})$ and the above result, we can find an element $\sum_{i,j} B_{ij}\theta^i \wedge \theta^j$ which is cohomologous to $\sum_{i,j} A_{ij}\theta^i \wedge \theta^j$, such that

$$d\tilde{\gamma} = \pi^*\Big(\sum_{i,j} B_{ij}\theta^i \wedge \theta^j\Big),$$

for some connection form $\tilde{\gamma}$ on P. Since $\sum_{i,j} B_{ij}\theta^i \wedge \theta^j$ and $\sum_{i,j} A_{ij}\theta^i \wedge \theta^j$ are cohomologous, there exists a 1-form α such that

$$d\alpha = \sum_{i,j} A_{ij}\theta^i \wedge \theta^j - \sum_{i,j} B_{ij}\theta^i \wedge \theta^j.$$

Next, we define a 1-form on P as

$$\gamma = \tilde{\gamma} + \pi^*\alpha.$$

Let us see that γ is a connection form on P that satisfies $d\gamma = \pi^*(\sum_{i,j} A_{ij}\theta^i \wedge \theta^j)$. Easily can we show the second assertion. In fact, we have

$$\begin{aligned} d\gamma &= d\tilde{\gamma} + d\pi^*\alpha \\ &= \pi^*\Big(\sum_{i,j} B_{ij}\theta^i \wedge \theta^j\Big) + \pi^*\Big(\sum_{i,j} A_{ij}\theta^i \wedge \theta^j - \sum_{i,j} B_{ij}\theta^i \wedge \theta^j\Big) \\ &= \pi^*\Big(\sum_{i,j} A_{ij}\theta^i \wedge \theta^j\Big). \end{aligned}$$

On the other hand, we have that γ is equivariant:

$$\begin{aligned} \gamma(R_{s*}u) = \tilde{\gamma}(R_{s*}u) + \pi^*\alpha(R_{s*}u) &= \tilde{\gamma}(u) + \alpha(\pi^*(R_{s*}u)) \\ &= (\gamma + \pi^*)(u) \\ &= \gamma(u). \end{aligned}$$

Finally, since $\pi^*\alpha$ is horizontal, we conclude that γ is a connection form. □

In the above proposition, we have found a connection form γ on P subject to

$$d\gamma = \pi^* \left(\sum_{i,j} A_{ij}\theta^i \wedge \theta^j \right),$$

where $\sum_{i,j} A_{ij}\theta^i \wedge \theta^j$ is an integral closed 2-form on M. However, as we will see next, we require for this 2-form to be a harmonic 2-form.

We have the following result from Kobayashi [18].

Proposition 3.3.5. *Given any positive number $\rho > 0$, there exits a harmonic 2-form $\sum_{i,j} A_{ij}\theta^i \wedge \theta^j$ on M, representing an element of $H^2_{dR}(M, \mathbb{Z})$, and a real number b such that*

$$\sum_{i,j=1}^{2n} |J_{ij} - bA_{ij}|^2 < \rho \text{ and } \sum_{i,j,k=1}^{2n} |bA_{ij;k}|^2 < \rho,$$

here the $A_{ij;k}$ are the components of the covariant derivative of A_{ij}.

Proof. By the theory of elliptic partial differential equations, there exists a positive constant C such that for all harmonic forms $\sum_{i,j} B_{ij}\theta^i \wedge \theta^j$ on M we have

$$\sum_{ijk} |B_{ij;k}|^2 \leq C \sum_{i,j} |B_{ij}|^2 .$$

Moreover, by cohomology Theory (see Warner [30]), we have that $H^2_{dR}(M, \mathbb{Z})$ form a basis in $H^2_{dR}(M, \mathbb{R})$. Then the set $\{b\tau; b \in \mathbb{R} \text{ and } \tau \in H^2_{dR}(M, \mathbb{Z})\}$ is dense in $H^2_{dR}(M, \mathbb{R})$. In other words, given a real number ρ_1, we can find a real number b and a 2-form $\sum_{i,j} A_{ij}\theta^i \wedge \theta^j$ in $H^2(M, \mathbb{Z})$, such that

$$\sum_{i,j} |bA_{ij} - J_{ij}|^2 < \rho_1.$$

By a Theorem of Hodge, we can choose $\sum_{i,j} A_{ij}\theta^i \wedge \theta^j$ so that this is a harmonic 2-form.

Taking $B_{ij} = bA_{ij} - J_{ij}$, we have that $\sum_{i,j} B_{ij}\theta^i \wedge \theta^j$ is a harmonic 2-form. So, here exists a positive constant C_1 such that

$$\sum_{i,j,k} |B_{ij;k}|^2 \leq C_1 \sum_{i,j} |B_{ij}|^2 \leq C_1\rho_1.$$

Because of $B_{ij} = bA_{ij} - J_{ij}$, we obtain $B_{ij;k} = bA_{ij;k} - J_{ij;k}$. Furthermore, by Remark 1.5.4, we have $\nabla J = 0$. By Remark 3.1.10, it implies that

$B_{ij;k} = bA_{ij;k}$. Finally, taking $\rho_1 = \dfrac{\rho}{C_1} <$ we arrive to

$$\sum_{i,j,k=1} |bA_{ij;k}|^2 < \rho.$$

$\square$

Suppose that the Kählerian pinching of M is greater than $4a^2$, where $0 < a < \frac{1}{2}$. Recall that by definition there exists a number $L > 0$ such that $4a^2 L < K^*(q) \leq L$.

Since we can normalize the metric g scaling by L, the above inequalities simplify to

$$4a^2 < K^*(q) \leq 1. \tag{3.17}$$

Remark 3.3.6. On a complete Kähler manifold M, if the Kählerian pinching is greater than δ, then the Riemannian pinching is greater than $\frac{\delta}{4}$. In fact, by definition, there exists $L > 0$ such that

$$\delta L < K^*(q) \leq L,$$

for all 2-plane q. As $K^*(q) = \dfrac{4K(q)}{1 + 3\cos^2\alpha(q)}$, then we have

$$\delta L < \frac{4K(q)}{1 + 3\cos^2\alpha(q)} \leq L.$$

Multiplying both sides of the inequalities by $\dfrac{1 + 3\cos^2\alpha(q)}{4}$, we obtain

$$\left(\frac{1 + 3\cos^2\alpha(q)}{4}\right) \delta L < K(q) \leq L \left(\frac{1 + 3\cos^2\alpha(q)}{4}\right).$$

As $\dfrac{1}{4} \leq \dfrac{(1 + 3\cos^2\alpha(q))}{4} \leq 1$, the above inequalities boil down to

$$\frac{\delta}{4} L < K(q) \leq L.$$

Thus, we conclude that the Riemannian pinching on M should be greater than $\frac{\delta}{4}$.

Returning to Inequality (3.17) and using Remark 3.3.6, we have them

$$a^2 < K(q) \leq 1.$$

Using a theorem of Myers (see Berger [2]), as M is complete and its sectional curvature is bounded, we deduce that M is compact.

Consider the metric $g = ds^2 = \sum_{j=1}^{n} (\theta^j)^2$ on M, and the tensor J expressed as (J_{ij}) with respect to orthonormal frame $\{E^1, ..., E^n\}$. Let p be a 2-plane in T_xP , and $\{X, Y\}$ an orthonormal basis for p, which we write as

$$X = \sum_{a=0}^{n} X^a \xi^a \quad \text{and} \quad Y = \sum_{a=0}^{n} Y^a \xi^a,$$

where $\{\xi^0, ..., \xi^n\}$ is an orthonormal frame with respect to the metric $d\sigma^2 = \pi^* ds^2 + (ab\gamma)^2$ on P subject to $\pi_*(\xi^i) = E^i$. Then, we have that the sectional curvature $R(p)$ of plane p is given by

$$R(p) = R(X, Y, X, Y) = \sum_{\alpha\beta\lambda\mu} R_{\alpha\beta\lambda\mu} X^\alpha Y^\beta X^\lambda Y^\mu.$$

On the other hand, let us suppose that the vectors $\tilde{X}, \tilde{Y}$ defined as $\tilde{X} = \sum_{i=1}^{n} X^i E^i$ and $\tilde{Y} = \sum_{i=1}^{n} Y^i E^i$ span a 2-plane q on $T_{\pi(x)}M$. In such case, the sectional curvature of q is given by

$$K(q) = \sum_{i,j,k,l=1}^{n} K_{ijkl} X^i Y^j X^k Y^l.$$

The numbers $K(q)$, $\bar{K}(q)$ and $K^*(q)$ are related by the equality

$$K^*(q) = \frac{K(q)}{\bar{K}(q)},$$

where $\bar{K}(q) = \dfrac{1 + 3\cos^2 \alpha(q)}{4}$. Then we can rewrite the corresponding inequalities of Proposition 3.2.5, as

$$4a^2 \leq K^*(q) \leq 1.$$

Hence using Inequality (3.17) and Proposition 3.2.5, we obtain

$$a^2 \leq S(p) \leq 1 - 3a^2.$$

Taking $\delta = 4a^2$ in the above expression, we arrive to

$$\frac{\delta}{4} \leq S(p) \leq 1 - \frac{3\delta}{4}.$$

This last relation is equivalent to

$$\frac{\delta}{4-3\delta} \le S(p) \le 1.$$

As the metric $d\sigma^2$ is normalized, the above inequalities mean that the Riemannian pinching of P is greater than $\frac{\delta}{4-3\delta}$.

Now we are ready to proof Theorem 3.3.3. This is based on Kobayashi [18].

Proof of Theorem 3.3.3. Let K_{ijkl} and $S_{\alpha\beta\lambda\mu}$ be defined as before. By the previous remarks, for $\delta < K^*(q) < 1$, we have

$$\frac{\delta}{4-3\delta} < S(p) < 1.$$

Since M is compact, then the bundle of 2-planes is compact. Furthermore as the assignment $p \mapsto S(p)$ is continous, there exists a $s > 0$ such that

$$\frac{\delta}{4-3\delta} + s < S(p) < 1 - s. \tag{3.18}$$

By Proposition 3.2.6, for $s > 0$, there is a $\rho > 0$ such $\sum_{ij} |bA_{ij} - J_{ij}|^2 < \rho$ and $\sum_{ijk} |bA_{ij;k}|^2 < \rho$ imply $|R(p) - S(p)| < s$, where $A_{ij}, A_{ij;k}$ and the components $R_{\alpha\beta\lambda\mu}$ of R are as in Proposition 3.2.6.

By Proposition 3.3.5, we can choose the components A_{ij} so that the 2-form $\sum_{i,j} A_{ij}\theta^i \wedge \theta^j$ on M is harmonic and represents an element of $H^2(M, \mathbb{Z})$. Then, by Proposition 3.3.4, for this harmonic 2-form there is a principal S^1-bundle P on M and a connection form γ on P such that

$$d\gamma = \pi^*\Big(\sum_{i,j} A_{ij}\theta^i \wedge \theta^j\Big).$$

Finally, since $R_{\alpha\beta\gamma\mu}$ are the components of the curvature tensor R with respect to the metric $d\sigma^2 = \pi^*(ds^2) + (ab\gamma)^2$ on P, the number $R(p)$ represents the sectional curvature of P.

Then, as we have $|R(p) - S(p)| < s$ and $S(p)$ verifies the inequality (3.18), we obtain

$$\frac{\delta}{4-3\delta} < S(p) - s < R(p) < S(p) + s < 1.$$

This last property implies that the sectional curvature of P is strictly greater than $\frac{\delta}{4-3\delta}$ and the theorem is settle. $\square$

Chapter 4

Einstein Metrics on Principal S^1-bundles

In this chapter we will construct Einstein metrics on the principal S^1-bundle following the method of Kobayashi. Afterwards we will exhibit explicitly this construction for the Hopf fibration. Moreover, as we will see in the last section, under certain conditions the method of S. Kobayashi allows us to construct Sasaki-Einstein metrics on the total space of a Boothby-Wang fibration.

4.1 The method of Kobayashi

Let (M, g) be a Riemannian manifold of dimension n. Given an orthonormal basis $\{E^1, ..., E^n\}$ of T_xM, we define the *Ricci curvature* of (M, g) by

$$Ric(X, Y) = \sum_{k=1}^{n} R(X, E^i, Y, E^i),$$

where X, Y are arbitrary vectors in T_xM. Also, the *scalar curvature* of (M, g) is given by

$$Scal = \sum_{k=1}^{n} Ric(E^k, E^k).$$

We say that (M, g) is an *Einstein manifold* with *Einstein constant* λ if

$$Ric(X, Y) = \lambda g(X, Y)$$

The *Ricci form* ρ_M is given by

$$\rho_M(X, Y) = Ric(JX, Y).$$

Theorem 4.1.1. *Let (M, g, J) be a Kähler-Einstein manifold with positive scalar curvature. Then we can construct a principal S^1-bundle over M and an Einstein metric with positive scalar curvature on P.*

Proof. Let J_{ij} be the components of J with respect to the orthonormal basis $\{\theta^1, ..., \theta^n\}$. Consider $\{E^1, ..., E^n\}$ an orthonormal basis of T_xM such that $\theta^i(E^j) = \delta_{ij}$. Then we can express a representative of the first Chern class $c_1(M)$, an element of $H^2_{dR}(M, \mathbb{Z})$, as $\frac{\lambda}{2\pi}\sum_{ij=1}^n J_{ij}\theta^i \wedge \theta^j$, for some constant λ. Indeed, by definition we have $c_1(M) = \frac{1}{2\pi}[\rho_M]$. Then, since g is an Einstein metric, we have

$$
\begin{aligned}
\rho_M(X, Y) = Ric(JX, Y) = \lambda g(JX, Y) &\\
&= \lambda \sum_{i=1}^n \theta^i(JX)\theta^i(Y)\\
&= \lambda \sum_{i=1}^n \theta^i(\sum_{j=1}^n (X^j J(E^j))\theta^i(Y)\\
&= \lambda \sum_{i=1}^n (\sum_{j=1}^n X^j\theta^i(\sum_{k=1}^n J_{kj}E^k))\theta^i(Y)\\
&= \lambda \sum_{i=1}^n (\sum_{j=1}^n X^j J_{ij})Y^i\\
&= \lambda \sum_{i,j=1}^n J_{ij}X^jY^i.
\end{aligned}
$$

Thus, we obtain an element $\frac{\lambda}{2\pi}\sum_{i,j=1}^n J_{ij}\theta^i \wedge \theta^j$ of the first Chern class, where λ is the constant given above. By Proposition 3.3.4, there is a principal S^1-bundle P and a connection γ on P such that

$$d\gamma = \pi^*(\frac{\lambda}{2\pi}\sum_{i,j=1}^n J_{ij}\theta^i \wedge \theta^j).$$

If we define the metric $d\sigma^2 = \pi^*(ds^2)+(2\pi a\lambda^{-1}\gamma)^2$ on P, by Proposition 3.1.11, the components $R_{\alpha\beta\lambda\mu}$ of the curvature form with respect to $d\sigma^2$ are given by

$$
\begin{aligned}
&R_{ijkl} = K_{ijkl} - a^2(2J_{ij}J_{kl} + J_{ik}J_{jl} - J_{il}J_{jk}),\\
&R_{i0k0} = -R_{i00k} = -R_{0ik0} = R_{0i0k} = a^2\delta_{ik},\\
&R_{\alpha\beta\lambda\mu} = 0, \qquad \text{otherwise.}
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
R_{ij} = \sum_{\lambda=0}^{n} R_{i\lambda j\lambda} &= \sum_{k=1}^{n} R_{ikjk} + R_{i0j0} \\
&= \sum_{k=1}^{n} (K_{ikjk} - a^2(2J_{ik}J_{jk} + J_{ij}J_{kk} - J_{ik}J_{kj})) + a^2\delta_{ij} \\
&= K_{ij} - a^2(2\delta_{ij} + \delta_{ij}) + a^2\delta_{ij} \\
&= K_{ij} - 2a^2\delta_{ij},
\end{aligned}
$$

Furthermore, we have

$$R_{i0} = \sum_{\lambda=0}^{n} R_{ik0k} + R_{i000} = 0$$

and

$$R_{00} = \sum_{\lambda=0}^{n} R_{0\lambda 0\lambda} = \sum_{k=1}^{n} R_{0k0k} + R_{0000} = \sum_{k=1}^{n} a^2 = na^2.$$

As $g = ds^2$ is an Einstein metric with Einstein constant λ, we have $K_{ij} = \lambda g_{ij} = \lambda\delta_{ij}$. Taking $a = (\frac{\lambda}{n+2})^{\frac{1}{2}}$, we obtain

$$R_{ij} = \lambda\delta_{ij} - \frac{2\lambda}{n+2}\,\delta_{ij} = \frac{n\lambda}{n+2}\,\delta_{ij}$$

and

$$R_{00} = \frac{n\lambda}{n+2}.$$

At the end we obtain $R_{\alpha\beta} = \dfrac{n\lambda}{n+2}\delta_{\alpha\beta}$, and so, the metric $d\sigma^2$ on P is Einstein, with positive scalar curvature. □

4.2 Sasakian manifolds

We give a very brief review of basic facts on Sasakian structure. For a thorough treatment on Sasakian Geometry see Boyer and Galicki [8] or Blair [6].

Let (M, φ, ξ, η) be an almost contact manifold of dimension $2n + 1$. Here we define the tensors N_1, N_2, N_3 and N_4 by

$$
\begin{aligned}
N_1(X, Y) &= [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] + 2d\eta(X, Y)\xi, \\
N_2(X, Y) &= (\mathsf{L}_{\varphi X}\eta)Y - (\mathsf{L}_{\varphi Y}\eta)X, \\
N_3(X) &= (\mathsf{L}_{\xi}\varphi)X, \\
N_4(X) &= (\mathsf{L}_{\xi}\eta)X.
\end{aligned}
$$

We say that the almost contact structure (φ, ξ, η) is *normal* if the tensors N_1, N_2, N_3 and N_4 vanish.

Remark 4.2.1. We have the usual identity for 2-forms

$$d\alpha(X, Y) = \frac{1}{2}(X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])).$$

In particular for η we obtain

$$d\eta(X, Y) = \frac{1}{2}(X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])).$$

Putting this in N_1 we get

$$N_1(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y]+ \\ (X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]))\xi.$$

Since $\varphi^2 = -id + \eta \otimes \xi$; that is $\varphi^2[X, Y] - \eta([X, Y])\xi = -[X, Y]$, when we replace in the above expression, this results in

$$N_1(X, Y) = [\varphi X, \varphi Y] - \varphi[X, \varphi y] - \varphi[\varphi X, Y] - [X, Y] +(X(\eta(Y)) - Y(\eta(X)))\xi. \tag{4.1}$$

Proposition 4.2.2. *For the tensors N_1, N_2, N_3 and N_4, defined above, we have that $N_1 = 0$ implies that N_2, N_3 and N_4 vanish.*

Proof. Making $Y = \xi$ in Equation 4.1, we have

$$N_1(X, \xi) = [\varphi X, \varphi\xi] - \varphi[X, \varphi\xi] - \varphi[\varphi X, \xi] - [X, \xi] + (X(\eta(\xi)) - \xi(\eta(X)))\xi.$$

From $\varphi(\xi) = 0$ and $\eta(\xi) = 1$, we have

$$N_1(X, \xi) = -\varphi[\varphi X, \xi] - [X, \xi] - \xi(\eta(X))\xi. \tag{4.2}$$

If $N_1 = 0$, we apply η in both sides in the above expression, in order to get

$$0 = -(\eta \circ \varphi)[\varphi X, \xi] - \eta([X, \xi]) - \xi(\eta(X))\eta(\xi).$$

From $\eta \circ \varphi = 0$ and $\eta(\xi) = 1$, we arrive at

$$0 = -\eta([X, \xi]) - \xi(\eta(X)).$$

In addition, as $(\xi\eta)X = \xi(\eta(X)) - \eta([\xi, X])$, we have

$$0 = -\eta([X, \xi]) - \xi(\eta(X)) = \eta([\xi, X]) - \xi(\eta(X)) = -(\mathsf{L}_\xi\eta)(X).$$

Thus, we achieve $N_4(X) = (\mathsf{L}_\xi\eta)(X) = 0$ for any vector field X.

To prove $N_3 = 0$, we consider Equation 4.2. As $N_1 = 0$, when we apply φ in (4.2) we arrive to

$$-\varphi^2[\varphi X, \xi] - \varphi[X, \xi] - \varphi(\xi(\eta(X))\xi) = 0.$$

Since $\varphi^2 = -id + \eta \otimes \xi$, we obtain

$$[\varphi X, \xi] - \eta([\varphi X, \xi])\xi - \varphi[X, \xi] - \varphi(\xi(\eta(X))\xi) = 0.$$

The antisymmetry of $[\cdot, \cdot]$, yields

$$-[\xi, \varphi X] + \eta([\xi, \varphi X]) + \varphi[\xi, X] - \xi(\eta(X))\varphi\xi = 0.$$

As $(\mathsf{L}_\xi\varphi)(X) = -\varphi[\xi, X] + [\xi, \varphi X]$ and $\varphi\xi = 0$, we get

$$-(\mathsf{L}_\xi\varphi)(X) + \eta([\xi, \varphi X]) = 0.$$

Because of $\eta \circ \varphi = 0$, we reach

$$(\mathsf{L}_\xi\eta)(\varphi X) = \xi(\eta(\varphi X)) - \eta([\xi, \varphi X]) = -\eta([\xi, \varphi X]).$$

As $(\mathsf{L}_\xi\eta)(\varphi X) = 0$, then $\eta([\xi, \varphi X]) = 0$. Therefore $N_3(X) = (\mathsf{L}_\xi\varphi)X = 0$ for any vector fields X.

On the other hand, to show that $N_1 = 0$ implies $N_2 = 0$, we use Equation 4.1:

$$N_1(X, Y) = [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] - [X, Y] + (X(\eta(Y)) - Y(\eta(X)))\xi.$$

Since $N_1 = 0$ we apply η in both sides and we obtain

$$\begin{aligned} 0 = {} & \eta([\varphi X, \varphi Y]) - (\eta \circ \varphi)[X, \varphi Y] - (\eta \circ \varphi)[\varphi X, Y] - \eta([X, Y]) + \\ & (X(\eta(Y)) - Y(\eta(X)))\eta(\xi). \end{aligned}$$

Since $\eta \circ \varphi = 0$ and $\eta(\xi) = 1$, we have

$$0 = \eta([\varphi X, \varphi Y]) - \eta([X, Y]) + X(\eta(Y)) - Y(\eta(X)).$$

Replacing X by φX in the above expression, and using the identities $\varphi^2 = -id + \eta \otimes \xi$ and $\eta \circ \varphi = 0$, we obtain

$$\begin{aligned} 0 &= \eta([\varphi^2 X, \varphi Y]) - \eta([\varphi X, Y]) + \varphi X(\eta(Y)) - Y(\eta(\varphi X)) \\ &= \eta([-X + \eta(X)\xi, \varphi Y]) - \eta([\varphi X, Y]) + \varphi X(\eta(Y)) \\ &= -\eta([X, \varphi Y]) + \eta([\eta(X)\xi, \varphi Y]) - \eta([\varphi X, Y]) + \varphi X(\eta(Y)) \\ &= -\eta([X, \varphi Y]) + \eta(\eta(X)[\xi, \varphi Y] - \varphi Y(\eta(X))\xi) - \eta([\varphi X, Y]) + \varphi X(\eta(Y)) \\ &= -\eta([X, \varphi Y]) + \eta(X)\eta([\xi, \varphi Y]) - \varphi Y(\eta(X))\eta(\xi) - \eta([\varphi X, Y]) + \varphi X(\eta(Y)). \end{aligned}$$

As $(\mathsf{L}_\xi\varphi)Y = [\xi, \varphi Y] - \varphi[\xi, Y] = 0$, then $[\xi, \varphi Y] = \varphi[\xi, Y]$. Replacing this above and using the identity $\eta(\xi) = 1$, we obtain

$$
\begin{aligned}
0 &= -\eta([X, \varphi Y]) + \eta(X)\eta \circ \varphi[\xi, Y] - \varphi Y(\eta(X)) - \eta([\varphi X, Y]) + \varphi X(\eta(Y)) \\
&= -\eta([X, \varphi Y]) - \varphi Y(\eta(X)) - \eta([\varphi X, Y]) + \varphi X(\eta(Y)) \\
&= \varphi X(\eta(Y)) - \eta([\varphi X, Y]) - (\varphi Y(\eta(X)) - \eta([\varphi Y, X])) \\
&= (\mathsf{L}_{\varphi X}\eta)Y - (\mathsf{L}_{\varphi Y}\eta)X \\
&= N_2(X, Y).
\end{aligned}
$$

Therefore $N_2(X, Y) = 0$ for all X, Y. □

Proposition 4.2.3. *Let (φ, ξ, η, g) be a contact metric structure on M. Then the tensors N_2 and N_4 vanish. Furthermore we have $N_3 = 0$ if and only if ξ is a Killing vector field.*

Proof. First we prove $N_2 = 0$. Since $(\mathsf{L}_{\varphi X}\eta)Y = \varphi X(\eta(Y)) - \eta([\varphi X, Y])$ and $d\eta(\varphi X, Y) = \frac{1}{2}(\varphi X(\eta(Y)) - Y(\eta(\varphi X)) - \eta([\varphi X, Y]))$, we arrive to $(\mathsf{L}_{\varphi X}\eta)Y = 2d\eta(\varphi X, Y) + Y(\eta(\varphi X))$. Because of $\eta \circ \varphi = 0$, we obtain $(\mathsf{L}_{\varphi X}\eta)Y = 2d\eta(\varphi X, Y)$. Analogously, we have $(\mathsf{L}_{\varphi Y}\eta)X = 2d\eta(\varphi Y, X)$. Thus, we get

$$N_2(X, Y) = (\mathsf{L}_{\varphi X}\eta)Y - (\mathsf{L}_{\varphi Y}\eta)X = 2d\eta(\varphi X, Y) - 2d\eta(\varphi Y, X).$$

As $d\eta(\varphi X, Y) = g(\varphi X, \varphi Y)$, then

$$N_2 = 2d\eta(\varphi X, Y) - 2d\eta(\varphi Y, X) = 2g(\varphi X, \varphi Y) - 2g(\varphi Y, \varphi X) = 0.$$

On the other hand, to verify $N_4 = 0$, we remember the equalities $(\mathsf{L}_\xi\eta)X = \xi(\eta(X)) - \eta([\xi, X])$ and $2d\eta = \xi(\eta(X)) - X(\eta(\xi)) - \eta([\xi, X])$. As $\eta(\xi) = 0$, then $(\mathsf{L}_\xi\eta)X = 2d\eta(\xi, X)$. Because of $d\eta(\xi, X) = g(\xi, \varphi X)$, we obtain

$$N_4(X) = (\mathsf{L}_\xi\eta)X = 2g(\xi, \varphi X) = 2(g(\varphi\xi, \varphi^2 X) - \eta(\xi)\eta(\varphi X) = 0.$$

Finally, to show that $N_3 = 0$ is equivalent to ξ being a Killing vector field, we remember that it is already equivalent to prove $\mathsf{L}_\xi g = 0$. Knowing $\mathsf{L}_\xi d\eta = 0$ to be true, we have

$$
\begin{aligned}
0 = (\mathsf{L}_\xi d\eta)(X, Y) &= \xi(d\eta(X, Y)) - d\eta([\xi, X], Y) - d\eta(X, [\xi, Y]) \\
&= \xi(g(X, \varphi Y)) - g([\xi, X], \varphi Y) - g(X, \varphi[\xi, Y]).
\end{aligned}
$$

On the other hand, we have

$$(\mathcal{L}_\xi g)(X, \varphi Y) = \xi(g(X, \varphi Y)) - g([\xi, X], \varphi Y) - g(X, [\xi, \varphi Y]),$$

and so

$$\xi(g(X, \varphi Y)) - g([\xi, X], \varphi Y) = (\mathcal{L}_\xi g)(X, \varphi Y) + g(X, [\xi, \varphi Y]).$$

Replacing the above expression in $(\mathcal{L}_\xi d\eta)(X, Y) = 0$, we obtain

$$0 = (\mathcal{L}_\xi d\eta)(X, Y) = (\mathcal{L}_\xi g)(X, \varphi Y) + g(X, [\xi, \varphi Y]) - g(X, \varphi[\xi, Y]).$$

As $N_3(Y) = (\mathcal{L}_\xi \varphi)Y = [\xi, \varphi Y] - \varphi[\xi, Y]$, we conclude

$$0 = (\mathcal{L}_\xi d\eta)(X, Y) = (\mathcal{L}_\xi g)(X, \varphi Y) + g(X, N_3(Y)).$$

Therefore we have $\mathcal{L}_\xi g = 0$ if and only if $N_3 = 0$. □

Definition 4.2.4. We say that an almost contact metric structure (φ, ξ, η, g) on M is *Sasakian* if this is a metric contact structure and (φ, ξ, η) is a normal structure on M. In this case we say that $(M; \varphi, \xi, \eta, g)$ is a *Sasakian manifold.*

We use the following theorem to establish the existence of a Sasakian structure on S^{2n+1}. Before, we state a lemma that is used in the proof.

Lemma 4.2.5. *Let (φ, ξ, η, g) be an almost contact metric structure on M. Then the convariant derivative of φ satisfies*

$$\begin{aligned} 2g((\nabla_X \varphi)Y, Z) = 3d\Omega(X, \varphi Y, \varphi Z) - 3d\Omega(X, Y, Z) + g(N_1(Y, Z), \varphi X) + \\ N_2(Y, Z)\eta(X) + 2d\eta(\varphi Y, X)\eta(Z) - 2d\eta(\varphi Z, X)\eta(Y). \end{aligned}$$

Proof. The proof of this Lemma is obtained for straightforward calculation. See Blair [6]. □

Theorem 4.2.6. *A manifold M with almost contact metric structure (φ, ξ, η, g) is Sasakian if and only if*

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X$$

holds.

Proof. Suppose (φ, ξ, η, g) is a Sasakian structure. Since (φ, ξ, η, g) is an almost contact metric structure, by the above lemma we have

$$2g((\nabla_X\varphi)Y, Z) = 3d\Omega(X, \varphi Y, \varphi Z) - 3d\Omega(X, Y, Z) + g(N_1(Y, Z), \varphi X) + N_2(Y, Z)\eta(X) + 2d\eta(\varphi Y, X)\eta(Z) - 2d\eta(\varphi Z, X)\eta(Y).$$

As (φ, ξ, η, g) is a Sasakian structure, this is a contact metric structure on M. It implies $\Omega = d\eta$. Moreover, (φ, ξ, η, g) is normal, which means $N_1 = 0$ and $N_2 = 0$. All together implies

$$\begin{aligned}
2g((\nabla_X\varphi)Y, Z) &= 2d\eta(\varphi Y, X)\eta(Z) - 2d\eta(\varphi Z, X)\eta(Y) \\
&= 2\Omega(\varphi Y, X)\eta(Z) - 2\Omega(\varphi Z, X)\eta(Y) \\
&= 2g(\varphi Y, \varphi X)\eta(Z) - 2g(\varphi Z, \varphi X)\eta(Y) \\
&= 2(g(X, Y) - \eta(X)\eta(Y))\eta(Z) - 2(g(Z, X) - \eta(Z)\eta(X))\eta(Y) \\
&= 2g(X, Y)\eta(Z) - 2g(Z, X)\eta(Y).
\end{aligned}$$

As $g(Z, \xi) = \eta(Z)$, then we can write

$$\begin{aligned}
g((\nabla_X\varphi)Y, Z) &= g(X, Y)g(Z, \xi) - g(Z, X)\eta(Y) \\
&= g(g(X, Y)\xi, Z) - g(\eta(Y)X, Z) \\
&= g(g(X, Y)\xi - \eta(Y)X, Z),
\end{aligned}$$

or what is the same

$$(\nabla_X\varphi)Y = g(X, Y)\xi - \eta(Y)X.$$

To show the converse, we suppose $(\nabla_X\varphi)Y = g(X, Y)\xi - \eta(Y)X$. Let us see that (φ, ξ, η, g) is a Sasakian structure on M. First, we show that (φ, ξ, η, g) is a metric contact structure on M. For this we must verify the equality $d\eta(X, Y) = g(X, \varphi Y)$. Since $\eta(X) = g(X, \xi)$, for the 2-form η we have

$$\begin{aligned}
d\eta(X, Y) &= \frac{1}{2}(X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y])) \\
&= \frac{1}{2}(X(g(Y, \xi) - Y(g(X, \xi)) - g([X, Y], \xi)).
\end{aligned}$$

As $X(g(Y, \xi)) = g(\nabla_X Y, \xi) + g(Y, \nabla_X\xi)$ and $[X, Y] = \nabla_X Y - \nabla_Y X$, we then obtain

$$\begin{aligned}
d\eta(X, Y) &= \frac{1}{2}(g(\nabla_X Y, \xi) + g(Y, \nabla_X\xi) - g(\nabla_Y X, \xi) - g(X, \nabla_Y\xi) - g(\nabla_X Y, \xi) \\
&\quad + g(\nabla_Y X, \xi)) \\
&= \frac{1}{2}(g(Y, \nabla_X\xi) - g(X, \nabla_Y\xi)).
\end{aligned}$$

On the other hand, putting $Y = \xi$ and using the equality $\eta(\xi) = 1$ in our hypothesis we get

$$(\nabla_X \varphi)\xi = g(X, \xi)\xi - \eta(\xi)X = \eta(X)\xi - X = \varphi^2 X.$$

Since $0 = \nabla_X \varphi\xi = (\nabla_X \varphi)\xi + \varphi\nabla_X \xi$, we reach $\varphi^2 X = -\varphi\nabla_X \xi$ for all X. Therefore we have $\varphi X = -\nabla_X \xi$. Replacing this above and using the skew-symmetry of Ω we arrive to

$$\begin{aligned} d\eta(X, Y) &= \frac{1}{2}(g(Y, \nabla_X \xi) - g(X, \nabla_Y \xi)) \\ &= \frac{1}{2}(-g(Y, \varphi X) + g(X, \varphi Y)) \\ &= \frac{1}{2}(-\Omega(Y, X) + \Omega(X, Y)) \\ &= \frac{1}{2}(\Omega(X, Y) + \Omega(X, Y)) \\ &= \Omega(X, Y). \end{aligned}$$

To check that (φ, ξ, η) is a normal structure, it is sufficient to know that N_1 vanishes. We have

$$N_1(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] + 2d\eta(X, Y)\xi.$$

As $[X, Y] = \nabla_X Y - \nabla_Y X$, then we obtain

$$\begin{aligned} N_1(X, Y) = {} & \nabla_{\varphi X}\varphi Y - \nabla_{\varphi Y}\varphi X + \varphi^2(\nabla_X Y - \nabla_Y X) - \varphi(\nabla_X \varphi Y - \nabla_{\varphi Y} X) \\ & - \varphi(\nabla_{\varphi X} Y - \nabla_Y \varphi X) + 2d\eta(X, Y)\xi \\ = {} & (\nabla_{\varphi X}\varphi)Y + \varphi\nabla_{\varphi X} Y - (\nabla_{\varphi Y}\varphi)X - \varphi\nabla_{\varphi Y} X + \varphi^2\nabla_X Y - \varphi^2\nabla_Y X \\ & - \varphi((\nabla_X \varphi)Y + \varphi\nabla_X Y - \nabla_{\varphi Y} X) - \varphi(\nabla_{\varphi X} Y - (\nabla_Y \varphi)X - \varphi\nabla_Y X) \\ & + 2d\eta(X, Y)\xi \\ = {} & (\nabla_{\varphi X}\varphi)Y - (\nabla_{\varphi Y}\varphi)X - \varphi(\nabla_X \varphi)Y + \varphi(\nabla_Y \varphi)X + 2d\eta(X, Y)\xi. \end{aligned}$$

On the other hand, by hypothesis we have

$$(\nabla_{\varphi X}\varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X$$

and

$$\varphi(\nabla_X \varphi)Y = \varphi(g(X, Y)\xi - \eta(Y)X) = -\eta(Y)\varphi X.$$

Thus, $(\nabla_{\varphi X}\varphi)Y - \varphi(\nabla_X\varphi)Y = g(\varphi X, Y)\xi$ holds. In a similar way we obtain $(\nabla_{\varphi Y}\varphi)X - \varphi(\nabla_Y\varphi)X = g(\varphi Y, X)\xi$. Replacing this in the above result, we finally obtain

$$\begin{aligned}
N_1(X, Y) &= (\nabla_{\varphi X}\varphi)Y - \varphi(\nabla_X\varphi)Y - ((\nabla_{\varphi Y}\varphi)X - \varphi(\nabla_Y\varphi)X) + 2d\eta(X, Y)\xi \\
&= g(\varphi X, Y)\xi - g(\varphi Y, X)\xi + 2d\eta(X, Y)\xi \\
&= \Omega(Y, X)\xi - \Omega(X, Y)\xi + 2d\eta(X, Y)\xi \\
&= -2\Omega(X, Y)\xi + 2d\eta(X, Y)\xi \\
&= 0
\end{aligned}$$

□

We next exhibit S^{2n+1} as an example of Sasakian manifold. For this, first we endow S^{2n+1} with an almost contact metric structure.

Proposition 4.2.7. *The sphere S^{2n+1} is endowed with an almost contact metric structure.*

Proof. Consider $\langle\cdot, \cdot\rangle$ to be the metric induced on S^{2n+1} by the standard dot product $\langle\cdot, \cdot\rangle$ of $\mathbb{R}^{2n+2}$ and J be the standard complex structure on $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$. Then we define a global vector field ξ by $J(N) = -\xi$, where N is the unitary vector field normal to S^{2n+1}. Notice that ξ is a global vector field tangent to S^{2n+1}. Indeed, we have

$$\langle N, \xi\rangle = \langle N, -J(N)\rangle = \langle J(N), -J^2(N)\rangle = \langle J(N), N\rangle = 0.$$

As a consequence, for each $X \in \mathfrak{X}(S^{2n+1})$ we can split the vector JX as

$$JX = \varphi(X) + \eta(X)N, \tag{4.3}$$

where $\varphi(X)$ is the tangent component of JX and $\eta(X)N$ its normal component.

We will show that $(\varphi, \eta, \xi, \langle\cdot, \cdot\rangle)$ is an almost contact metric structure on S^{2n+1}. Taking $X \in \mathfrak{X}(S^{2n+1})$, we have

$$J\varphi(X) = \varphi^2(X) + \eta(\varphi(X))N.$$

Since $\varphi(X)$ is tangent to S^{2n+1}, we get $\eta(\varphi(X)) = 0$. We obtain so

$$\varphi^2(X) = J\varphi(X) = J(JX - \eta(X)N) = -X - \eta(X)JN = -X + \eta(X)\xi.$$

Thus, we have $\varphi^2 = -id + \eta \otimes \xi$.

On the other hand, putting $X = \xi$ in Equation 4.3, we get $J\xi = \varphi(\xi)+\eta(\xi)N$. As $J\xi = N$ holds by definition, we obtain $\eta(\xi) = 1$ and $\varphi(\xi) = 0$. Finally, for the metric $(\cdot, \cdot)$, we have

$$\begin{aligned}(\varphi(X), \varphi(Y)) &= (JX - \eta(X)N, JY - \eta(Y)N) \\ &= (JX, JY) - \eta(X)(N, JY) - \eta(Y)(JX, N) + \eta(X)\eta(Y).\end{aligned}$$

But we also have $(N, JY) = (N, \varphi(Y) - \eta(Y)N) = -\eta(Y)$, and thus we get $(JX, N) = -\eta(X)$. Replacing in the above equality, we obtain

$$(\varphi(X), \varphi(Y)) = (JX, JY) - \eta(X)\eta(Y) = (X, Y) - \eta(X), \eta(Y).$$

In this way we have established that $(\varphi, \xi, \eta, (\cdot, \cdot))$ is an almost contact metric structure on S^{2n+1}. □

Next we show that S^{2n+1} together with the almost contact metric structure (φ, ξ, η, g) is a Sasakian manifold. For this we use the above proposition. We need previously a technical result.

Lemma 4.2.8. *(Gauss - Codazzi). Let $(M, (\cdot, \cdot))$ be an embedded submanifold in $(\mathbb{R}^{2n+2}, (\cdot, \cdot))$ of dimension $2n+1$, with the metric induced by the Euclidean metric. If we write as N the unit normal vector field of M, we have*

$$D_X Y = \nabla_X Y - (\cdot, \cdot)N,$$

where D is the covariant derivative on $\mathbb{R}^{2n+2}$ and ∇ is the covariant derivative on M.

Proof. See Kobayashi [17] or Falcitelli [10]. □

Remark 4.2.9. In the above lemma notice that $D_X N = X$ holds for all vector field on M.

Proposition 4.2.10. *Let (φ, ξ, η, g) be the almost contact metric structure on S^{2n+1} defined above. Then $(S^{2n+1}; \varphi, \xi, \eta, g)$ is a Sasakian manifold.*

Proof. By Theorem 4.2.6, we only need to verify the equality $(\nabla_X\varphi)Y = g(X, Y)\xi - \eta(Y)X$. For this, if we consider D as the covariant derivative in $\mathbb{R}^{2n+2}$ with respect to the Euclidean metric, we have

$$(D_X J)Y = D_X JY - JD_X Y.$$

We have $JY = \varphi Y + \eta(Y)N$ and $D_X Y = \nabla_X Y - g(X,Y)N$, we obtain

$$\begin{aligned}(D_X J)Y &= D_X(\varphi Y + \eta(Y)N) - J(\nabla_X Y - g(X,Y)N) \\ &= D_X\varphi Y + X(\eta(Y))N + \eta(Y)D_X N - J\nabla_X Y + J(g(X,Y)N).\end{aligned}$$

By Remark 4.2.9. we have $D_X N = X$. Moreover, applying Lemma 4.2.8 to $D_X\varphi Y$ we get

$$\begin{aligned}(D_X J)Y &= \nabla_X\varphi Y - g(X,\varphi Y)N + X(g(Y,\xi))N + \eta(Y)X - J\nabla_X Y \\ &\quad + g(X,Y)JN \\ &= (\nabla_X\varphi)Y + \varphi\nabla_X Y - g(X,\varphi Y)N + X(g(Y,\xi))N + \eta(Y)X \\ &\quad - J\nabla_X Y - g(X,Y)\xi.\end{aligned}$$

Since $\varphi\nabla_X Y = J\nabla_X Y - \eta(\nabla_X Y)N$, we arrive to

$$\begin{aligned}(D_X J)Y = (\nabla_X\varphi)Y - \eta(\nabla_X Y)N - g(X,\varphi Y)N + X(g(Y,\xi))N \\ + \eta(Y)X - g(X,Y)\xi\end{aligned}$$

As $(\mathbb{R}^{2n+2}, J)$ is a Kähler manifold, we have $D_X J = 0$. Thus we get

$$(\nabla_X\varphi)Y + \eta(Y)X - g(X,Y)\xi = 0$$

and

$$-\eta(\nabla_X Y)N - g(X,\varphi Y)N + X(g(Y,\xi))N = 0.$$

In other words (φ, ξ, η, g) is a Sasakian structure on S^{2n+1}. □

4.3 The Boothby-Wang fibration

Let (M,ω) be a symplectic manifold with $[\omega] \in H^2_{dR}(M,\mathbb{Z})$, i.e, ω is an integral symplectic form. Since $H^2_{dR}(M,\mathbb{Z}) \cong P(M,S^1)$, we have principal S^1-bundle over M, say $\pi : P \to M$, where the connection form γ on P satisfies $d\gamma = \pi^*\omega$. Furthermore, we can write $\omega = \sum_{i,j} A_{ij}\theta^i \wedge \theta^j$. The principal S^1-bundle $\pi : (P,\gamma) \to (M,\omega)$, with $d\gamma = \pi^*\omega$, is called the *Boothby - Wang fibration*.

Notice that the connection form γ establishes a contact structure on P. Indeed, since the Lie algebra of S^1 is $\mathbb{R}$, γ is a real-valued form. Moreover, if ξ is a vertical vector field such that $\gamma(\xi) = 1$ and $X_1, X_2, \ldots, X_{2n}$ are linearly independent horizontal vector fields such that $\theta^i(\pi(X_j)) = \delta_{ij}$, then we have

$$\begin{aligned}\gamma \wedge (d\gamma)^n(\xi, X_1, \ldots, X_{2n}) &= \gamma \wedge \pi^*\omega^n(\xi, X_1, \ldots, X_{2n}) \\ &= \gamma(\xi)\pi^*\omega^n(X_1, \ldots, X_{2n}).\end{aligned}$$

Since ω is symplectic, we have $\gamma \wedge (d\gamma)^n \neq 0$. Thus (P, γ) is a contact manifold. Besides, ξ is the Reeb vector field of the contact form γ.

Following the work of Hatakeyama [14], we have the proposition given below.

Proposition 4.3.1. *Let $\pi : (P, \gamma) \to (M, \omega)$ be a Boothby-Wang fibration, where $d\gamma = \pi^*\omega$. If ξ is the Reeb vector field defined as above and J is an almost complex structure on M, then there exists a $(1, 1)-$tensor φ on P such that (φ, ξ, γ) is an almost contact structure on P.*

Proof. Since the horizontal space $H_p \subset T_pP$ is isomorphic to $T_{\pi(p)}M$ for all $p \in P$, we can associate to each vector field X on M a unique vector field $\tilde{X}$ on P such that $\tilde{X}(p) \in H_p$ and $\pi_*\tilde{X}(p) = X(\pi(p))$. We write this isomorphism as $\tilde{\pi} : \mathfrak{X}(M) \to \mathfrak{X}(M)$, where $\tilde{\pi}(X) = \tilde{X}$. Then we define the (1, 1)-tensor φ on P as

$$\varphi X = \tilde{\pi} J \pi_* X.$$

Let us see that $\varphi^2 = -id + \eta \otimes \xi$ holds. For this, taking $X \in \mathfrak{X}(P)$, we have $X(p) = \tilde{\pi}\pi_*X(p) + \eta(X)\xi_p \in H_p \oplus V_p$. Then we obtain

$$\varphi^2 X = \varphi\tilde{\pi}J\pi_*X = \tilde{\pi}J\pi_*(\tilde{\pi}J\pi_*X).$$

As $\tilde{\pi}J\pi_*X(p) \in H_p$ for all $p \in P$, we have $\pi_*(\tilde{\pi}J\pi_*X) = J\pi_*X$. Thus we get

$$\varphi^2 X = \tilde{\pi}J(J\pi_*X) = \tilde{\pi}J^2\pi_*X = \tilde{\pi}(-\pi_*X) = -\tilde{\pi}\pi_*X = \eta(X)\xi - X,$$

as required. □

Next we impose an almost contact metric structure on the almost contact structure defined above.

Proposition 4.3.2. *Let $\pi : (P, \gamma) \to (M, \omega)$ be a Boothby-Wang fibration and (φ, ξ, γ) the almost contact structure defined in the proposition above. If g is a Riemannian metric on M compatible with the almost complex structure J, then $(\varphi, \xi, \gamma, \tilde{g})$ is a contact metric structure on P, where $\tilde{g}$ is defined by*

$$\tilde{g}(X, Y) = g(\pi_*X, \pi_*Y) + \gamma(X)\gamma(Y).$$

Furthermore, ξ is a Killing vector field with respect to the metric $\tilde{g}$. That is, the contact metric structure is K-contact.

Proof. At the beginning of Chapter 2 we have seen that if $\tilde{g}$ is defined in this way, then this is a metric. On the other hand, to show that $(\varphi, \xi, \gamma, \tilde{g})$ is an almost contact metric structure on P we only need to verify the equality

$$\tilde{g}(\varphi X, \varphi Y) = \tilde{g}(X, Y) - \gamma(X)\gamma(Y).$$

However, by definition we have

$$\tilde{g}(\varphi X, \varphi Y) = g(\pi_* \varphi X, \pi_* \varphi Y) + \gamma(\varphi X)\gamma(\varphi Y).$$

By Proposition 1.6.7, we have $\gamma \circ \varphi = 0$. Moreover, since $\pi_* \tilde{\pi} J \pi_* X = J \pi_* X$ and g is compatible with J, we obtain

$$\begin{aligned}
\tilde{g}(\varphi X, \varphi Y) &= g(\pi_* \varphi X, \pi_* \varphi Y) \\
&= g(\pi_* \tilde{\pi} J \pi_* X, \pi_* \tilde{\pi} J \pi_* Y) \\
&= g(J \pi_* X, J \pi_* Y) \\
&= g(\pi_* X, \pi_* Y) \\
&= \tilde{g}(X, Y) - \gamma(X)\gamma(Y).
\end{aligned}$$

Therefore $(\varphi, \xi, \gamma, \tilde{g})$ is an almost contact metric structure on P.
In addition, to show that $(\varphi, \xi, \gamma, \tilde{g})$ is a contact metric structure on P we must verify the equality $\tilde{g}(X, \varphi Y) = d\gamma(X, Y)$. By definition we have

$$\tilde{g}(X, \varphi Y) = g(\pi_* X, \pi_* \varphi Y) + \gamma(X)\gamma(\varphi Y) = g(\pi_* X, \pi_* \varphi Y).$$

Since $\pi_*(\tilde{\pi} J \pi_* Y) = J \pi_* Y$, we have $\pi_* \varphi Y = J \pi_* Y$. Thus, we arrive to

$$\tilde{g}(X, \varphi Y) = g(\pi_* X, J \pi_* Y) = \omega(\pi_* X, \pi_* Y) = d\gamma(X, Y).$$

Finally, since $\tilde{g} = \pi^* g + \gamma \otimes \gamma$, we obtain

$$\mathsf{L}_\xi \tilde{g} = \mathsf{L}_\xi \pi^* g + \mathsf{L}_\xi \gamma \otimes \gamma + \gamma \otimes \mathsf{L}_\xi \gamma.$$

As $\mathsf{L}_\xi \eta = 0$ and $\pi^* g$ is invariant under right action generated by ξ, we get $\mathsf{L}_\xi \tilde{g} = 0$. Therefore ξ is a Killing vector field and P is a K-contact manifold. $\square$

Proposition 4.3.3. *Given the Boothby-Wang fibration $\pi : (P, \gamma) \to (M, \omega)$ defined as above, the K-contact structure $(\varphi, \xi, \gamma, \tilde{g})$ on P defined in Proposition 4.3.2 is Sasakian if (M, J, g) is Kähler.*

Proof. By Proposition 4.2.2, we only need to show $N_1 = 0$. First, we see the case when X and Y are vertical vector field. It is sufficient to take $X = Y = \xi$. Then, using the formula of Remark 4.2.1, we have

$$N_1(\xi, \xi) = [\varphi\xi, \varphi\xi] - \varphi[\xi, \varphi\xi] - \varphi[\varphi\xi, \xi] - [\xi, \xi] - (\xi(\gamma(\xi)) - \xi(\gamma(\xi)))\xi = 0.$$

Suppose X is horizontal and $Y = \xi$. By the identity $\varphi\xi = 0$ we obtain

$$\begin{aligned} N_1(X, \xi) &= [\varphi X, \varphi\xi] - \varphi[X, \varphi\xi] - \varphi[\varphi X, \xi] - [X, \xi] - (X(\gamma(\xi)) - \xi(\gamma(X)))\xi \\ &= -\varphi[\varphi X, \xi] - [X, \xi]. \end{aligned}$$

Since $(\varphi, \xi, \eta, \tilde{g})$ is a K-contact structure, we have $N_3(X) = (\mathsf{L}_\xi\varphi)X = 0$. But this implies $(\mathsf{L}_\xi\varphi)X = [\xi, \varphi X] - \varphi[\xi, X] = 0$. Thus, $\varphi[\xi, X] = [\xi, \varphi X]$. Applying φ to both sides, we get $\varphi^2[\xi, X] = \varphi[\xi, \varphi X]$. As $\varphi^2 = -id + \gamma \otimes \xi$, we obtain $-[\xi, X] + \gamma([\xi, X])\xi = \varphi[\xi, \varphi X]$. Because ξ is vertical and X is horizontal, we have that $[\xi, X]$ is horizontal. Therefore $-[\xi, X] = \varphi[\xi, \varphi X]$. Replacing this above, we arrive to $N_1(X, \xi) = 0$.

If X and Y are horizontal vector field, we have $\gamma(X) = \gamma(Y) = 0$ and by Remark 4.2.1 we have

$$N_1(X, Y) = [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] - [X, Y] + (X(\gamma(Y)) - Y(\gamma(X)))\xi.$$

Furthermore, by definition we have $\varphi = \tilde{\pi} J\pi_*$. Thus, this can be rewritten as

$$\begin{aligned} N_1(X, Y) &= [\tilde{\pi} J\pi_* X, \tilde{\pi} J\pi_* Y] - \tilde{\pi} J\pi_*[X, \tilde{\pi} J\pi_* Y] - \tilde{\pi} J\pi_*[\tilde{\pi} J\pi_* X, Y] - [X, Y] \\ &= [\tilde{\pi} J\pi_* X, \tilde{\pi} J\pi_* Y] - \tilde{\pi} J[\pi_* X, \pi_*\tilde{\pi} J\pi_* Y] - \tilde{\pi} J[\pi_*\tilde{\pi} J\pi_* X, \pi_* Y] \\ &\quad - [X, Y]. \end{aligned}$$

As $\pi_*\tilde{\pi} J\pi_* X = J\pi_* X$, we arrive to

$$N_1(X, Y) = [\tilde{\pi} J\pi_* X, \tilde{\pi} J\pi_* Y] - \tilde{\pi} J[\pi_* X, J\pi_* Y] - \tilde{\pi} J[J\pi_* X, \pi_* Y] - [X, Y].$$

Remember that a vector field X vanishes if and only if we have $\pi_* X = 0$ and $\gamma(X) = 0$. Thus, $N_1(X, Y) = 0$ if and only if $\pi_* N_1(X, Y) = 0$ and $\gamma(N_1(X, Y)) = 0$. So we have

$$\begin{aligned} \pi_* N_1(X, Y) = \pi_*[\tilde{\pi} J\pi_* X, \tilde{\pi} J\pi_* Y] - \pi_*\tilde{\pi} J[\pi_* X, J\pi_* Y] - \\ \pi_*\tilde{\pi} J[J\pi_* X, \pi_* Y] - \pi_*[X, Y]. \end{aligned}$$

Since $\pi_* \tilde{\pi} X = X$ and $\pi_*[X, Y] = [\pi_* X, \pi_* Y]$ for all vector fields X and Y, we obtain

$$\begin{aligned}
\pi_* N_1(X, Y) &= [\pi_*\tilde{\pi} J \pi_* X, \pi_*\tilde{\pi} J \pi_* Y] - J[\pi_* X, J\pi_* Y] - \\
&\qquad J[J\pi_* X, \pi_* Y] - [\pi_* X, \pi_* Y] \\
&= [J\pi_* X, J\pi_* Y] - J[\pi_* X, J\pi_* Y] - J[J\pi_* X, \pi_* Y] - [\pi_* X, \pi_* Y] \\
&= N^J(\pi_* X, \pi_* Y).
\end{aligned}$$

Therefore $\pi_* N_1(X, Y) = 0$ is equivalent to $N^J(\pi_X, \pi_* Y) = 0$. This last happens precisely when J is a complex structure.
About the condition $\gamma(N_1(X, Y)) = 0$, we have

$$\gamma(N_1(X, Y)) = \gamma([\varphi X, \varphi Y]) - \gamma \circ \varphi[X, \varphi Y] - \gamma \circ \varphi[\varphi X, Y] - \gamma([X, Y]).$$

As $\gamma \circ \varphi = 0$, then we obtain

$$\gamma(N_1(X, Y)) = \gamma([\varphi X, \varphi Y]) - \gamma([X, Y]).$$

On the other hand, for the 1-form γ we have

$$d\gamma(X, Y) = \frac{1}{2}[X(\gamma(Y)) - Y(\gamma(X)) - \gamma([X, Y])).$$

Thus, we have $\gamma([X, Y]) = -2d\gamma(X, Y)$. Analogously, we obtain $\gamma([\varphi X, \varphi Y]) = -2d\gamma(\varphi X, \varphi Y)$. Since $d\gamma = \pi^*\omega$, we arrive to

$$\begin{aligned}
\gamma(N_1(X, Y)) &= -2\pi^*\omega(\varphi X, \varphi Y) + 2\pi^*\omega(X, Y) \\
&= -2\omega(\pi_*\varphi X, \pi_*\varphi Y) + 2\omega(\pi_* X, \pi_* Y) \\
&= -2\omega(J\pi_* X, J\pi_* Y) + 2\omega(\pi_* X, \pi_* Y).
\end{aligned}$$

Then $\gamma(N_1(X, Y)) = 0$ if and only if $\omega(J\pi_* X, J\pi_* Y) = \omega(\pi_* X, \pi_* Y)$. This last condition happens when ω and J are compatible. Therefore we have $\pi_* N_1(X, Y) = 0$ and $\gamma(N_1(X, Y)) = 0$ if and only if (M, J, g) is a Kahler manifold. Thus, this is equivalent to $(P, \varphi, \xi, \gamma, \tilde{g})$ being a Sasakian manifold. □

As we will see next, an interesting example of a Boothby-Wang fibration is the Hopf fibration $\pi : S^{2n+1} \to \mathbb{CP}^n$. Before working this over we give a definition equivalent to a connection 1-form on principal S^1-bundles.

Proposition 4.3.4. *Let $P \to M$ be a principal S^1-bundle with S^1-action on P generated by a vector field ξ of period 2π. A 1-form γ is a connection 1-form over P if it verifies*

(1) $\mathcal{L}_\xi \gamma = 0$,

(2) $\gamma(\xi) = 1$.

Proof. Taking ξ as the vertical vector field on the fibration $\pi : P \to M$, we have by (1) that γ is invariant under the S^1-action generated by ξ. Moreover, since $\gamma(\xi) = 1$, we obtain by Remark 2.1.10 that γ is a connection form on P □

Example 4.3.5. [Hopf fibration] We consider the principal S^1-bundle $\pi : S^{2n+1} \to \mathbb{CP}^n$ defined in Example 2.1.3. On $\mathbb{CP}^n$ we have the complex coordinates given in Example 1.4.2. Recall that they are the open subsets $U_k = \{[z_0 : ... : z_n]/z_k \neq 0\}$ with coordinate maps $\varphi_k : U_k \to \mathbb{C}^n$ defined by

$$\varphi_k([z_0 : ... : z_n]) = \left(\frac{z_0}{z_k}, ..., \frac{z_{k-1}}{z_k}, \frac{z_{k+1}}{z_k}, ..., \frac{z_n}{z_k}\right).$$

Since $\mathbb{C}^n \cong \mathbb{R}^{2n}$, we freely obtain real coordinates $\varphi_k = (x_1, y_1, ..., x_n, y_n)$ on $\mathbb{CP}^n$. On the other hand, we have defined in Example 1.5.6 a metric on $\mathbb{CP}^n$, the Fubini-Study metric. Taking the real coordinates given above, we express the Kähler form of this metric as

$$\omega_k = \frac{1}{\pi}\frac{\sum_{l=1}^n dx_l \wedge dy_l}{1+\sum_{j=1}^n (x_j)^2+(y_j)^2} + \frac{1}{\pi}\frac{\sum_{k,l=1}^n x_k y_l dx_k \wedge dx_l + \sum_{k,l=1}^n x_k y_l dy_k \wedge dy_l - \sum_{k,l=1}^n (x_k x_l + y_k y_l) dx_k \wedge dy_l}{(1+\sum_{j=1}^n (x^j)^2+(y^j)^2)^2},$$

on each U_K. Using the real coordinates $\varphi_k = (x_1, y_1, ..., x_n, y_n)$ and the trivialization ψ_k of the principal S^1-bundle $\pi : S^{2n+1} \to \mathbb{CP}^n$ given in Example 2.1.3, we obtain real coordinates on S^{2n+1} given by

$$\tilde{\psi}_k = (\varphi_k \circ \pi, t) = (x_1 \circ \pi, y_1 \circ \pi, ..., x_n \circ \pi, y_n \circ \pi, t).$$

Writing $x_i = x_i \circ \pi$ and $y_i = y_i \circ \pi$, we define the 1-form γ on S^{2n+1} as

$$\gamma = \sum_i \frac{x_i dy_i - y_i dx_i}{2\pi(1+\sum_l x_l^2 + y_l^2)} + dt.$$

From a direct calculation we obtain $d\gamma = \pi^*\omega_{FS}$, where ω_{FS} is the Kähler form for the Fubini-Study metric. Since $\gamma \wedge (d\gamma)^n = \gamma \wedge \pi^*\omega^n{}_{FS}$ and ω_{FS} is a symplectic form, we have that γ defines a contact structure on S^{2n+1}. For this contact form γ, its Reeb vector field is given by $\xi = \frac{\partial}{\partial t}$. Moreover, because of $L_\xi \gamma = 0$, we obtain that γ is a connection form on S^{2n+1}. Therefore, $\pi : (S^{2n+1}, \gamma) \to (\mathbb{CP}^n, \omega_{FS})$ is a Boothby-Wang fibration. Finally, using the above propositions, we obtain a Sasakian structure $(\varphi, \xi, \gamma, \tilde{g})$ on S^{2n+1}, with $\tilde{g}$ given by $\tilde{g} = \pi^* g_{FS} + \gamma \otimes \gamma$. If we want an Einstein metric, we should multiply the connection form γ by a constant b. Then we obtain another metric $g' = \pi^*(b g_{FS}) + b^2 \gamma \otimes \gamma$ on S^{2n+1}. By the method of Kobayashi, we choose $b = a\lambda^{-1}$, where $\lambda = n + 1$ and $a = \left(\frac{\lambda}{n+2}\right)^{\frac{1}{2}}$. Therefore, S^{2n+1} together with g' and the connection form $b\gamma$ defines a Sasaki-Einstein manifold.

Finally, using the same technique, we can establish the following result.

Proposition 4.3.6. *Let $\pi : P \to M$ be a Boothby-Wang fibration. If M is a complete Kähler manifold with positive sectional curvature, then we can construct an associated Sasakian-Einstein structure on P.*

References

[1] Ballmann, W. *Lectures on Kähler manifolds.* ESI Lectures in Mathematics & Physics. European Mathematical Society. (2006).

[2] Berger, M. *A panoramic view of Riemannian geometry.* Springer-Verlag Berlin Heidelberg. (2003).

[3] Besse, A. *Einstein manifolds.* Springer-Verlag, New York. (1987).

[4] Bishop, R. *Riemannian Geometry.* University of Illinois at Urbana-Champaign. (2013).

[5] Blair, D. *Contact manifolds in Riemannian geometry.* Springer-Verlag Berlin- Heidelberg- New York. (1976).

[6] Blair, D. *Riemannian geometry of contact and symplectic manifolds.* Springer Science+Business Media, New York. (2010).

[7] Boyer, C., Galicki, K. *Einstein metrics on sphere.* Ann. of Math., 162, 557-580. (2005)

[8] Boyer, C., Galicki, K. *Sasakian Geometry.* Oxford University Press Inc., New York. (2008).

[9] Cannas da Silva, A. *Lectures on symplectic geometry.* Springer-Verlag Berlin Heidelberg. (2001).

[10] Falcitelli, M., Ianus, S., Pastore, A. *Riemannian submersions.* World

Scientific, River Edge, NJ, USA. (2004).

[11] Figueroa, J. *Gauge theory. Lecture notes from a course on gauge theory.* University of Edinburgh. (2006). Retrieved from https://empg.maths.ed.ac.uk/Activities/GT/

[12] Geiges, H. *An introduction to contact topology.* Cambridge University Press Inc., New York. (2008).

[13] Griffiths, P., Harris, J. *Principles of algebraic geometry.* Vol. I, A Wiley- Interscience publication, New-York. (1978).

[14] Hatakeyama, Y. *Some notes on differentiable manifolds with almost contact structures.* Tôhoku Math. J.(2), 15, 176-181. (1963).

[15] Huybrechts, D. *Complex geometry. An introduction.* Springer-Verlag Berlin Heidelberg. (2005).

[16] Klingenberg, W. *Riemannian geometry.* de Gruyter Studies in Mathematics, Berlin- New York. (1982).

[17] Kobayashi, S., Nomizu, K. *Foundations of differential geometry.* Vol. I,II, Interscience Publishers, New-York. (1963), (1969).

[18] Kobayashi, S. *Topology of positively pinched Kähler manifolds.* Tôhoku Math. J., 15, 121-139. (1963).

[19] Kobayashi, S. *Principal fibre bundles with the 1-dimensional toroidal group.* Tôhoku Math. J.(2), 8, 29-45. (1956).

[20] Lee, J. *Riemannian manifolds: An introduction to curvature.* Springer-Verlag, New York. (1997).

[21] McDuff, D., Salamon, D. *Introduction to symplectic Topology.* Oxford

University Press, New-York. (1999).

[22] Morita, S. *Geometry of differential forms (Translations of mathematical monographs).* American Mathematical Society. (2001)

[23] Moroianu, A. *Lectures on Kähler geometry.* Cambridge University Press, New York. (2007).

[24] Newlander, A., Nirenberg L. *Complex analytic coordinates in almost complex manifolds.* Annals of Mathematics, second series, Vol. 65, No. 3, pp. 391-404. (1957).

[25] Serre, Jean-Pierre. *Faisceaux Algebriques Coherents.* [*Coherent algebraic sheaves.*] Annals of Mathematics, second series, Vol. 61, No. 2, pp. 197-278 (1955) [Springer collected works in Mathematics. (2003)] .

[26] Simanca, S. *Canonical metrics on compact almost complex manifolds.* Publ. Mat. IMPA. Inst. Nac. Mat. Pura Apl. (IMPA), Rio de Janeiro. (2004)

[27] Tian, G. *Canonical metrics in Kähler geometry.* Lectures in mathematics ETH Zürich. Birkhäuser Verlag, Basel- Boston- Berlin. (2000).

[28] Trang, T. *Einstein metrics on principal circle bundles.* Thesis for the degree of Honours in Pure Mathematics. University of Adelaide (2014). Australia .

[29] Wang, J. *Einstein metrics on bundles.* Ph.D. Thesis. McMaster University (1996). Canada.

[30] Warner, F. *Foundation of differentiable manifolds and Lie groups.* Springer-Verlag, New York. (1983).